Chapter 6
Structures for Discrete-Time Systems
Outline

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6.0 Introduction

The difference equation, the impulse response, and the system function are equivalent characterizations of the input/output relation of a linear time-invariant discrete-time system.
Ex. A system function:

\[ H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a z^{-1}} \]

The impulse response of this system is:

\[ h[n] = b_0 a^n u[n] + b_1 a^{n-1} u[n-1] \]
The first order difference equation is:

\[ y[n] - ay[n-1] = b_0 x[n] + b_1 x[n-1] \]

or

\[ y[n] = ay[n-1] + b_0 x[n] + b_1 x[n-1] \]
The basic elements required for implementation of a linear time-invariant discrete-time system are:

![Block Diagram Representation of Linear Constant-Coefficient Difference Equations](image)
Example

\[ y[n] = a_1 y[n-1] + a_2 y[n-2] + b \times x[n] \]

The corresponding system function is

\[ H(z) = \frac{b}{1 - a_1 z^{-1} - a_2 z^{-2}} \]
The system block diagram is
Generalized to higher-order difference equation:

\[ y[n] - \sum_{k=1}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \]
Rewriting equation above

\[ y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \]

when

\[ v[n] = \sum_{k=0}^{M} b_k x[n-k] \]
Can be view as

\[
H(z) = H_2(z)H_1(z) = \left( \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) \left( \sum_{k=0}^{N} b_k z^{-k} \right)
\]

\[
V(z) = H_1(z)X(z) = \left( \sum_{k=0}^{N} b_k z^{-k} \right)X(z)
\]
\[ Y(z) = H_2(z)V(z) = \left( \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) V(z) \]
Can be represent as

\[
H(z) = H_1(z)H_2(z) = \left( \sum_{k=0}^{N} b_k z^{-k} \right) \left( \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right)
\]

\[
W(z) = H_2(z)X(z) = \left( \frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right) X(z)
\]
\[ Y(z) = H_1(z)W(z) = \left( \sum_{k=0}^{N} b_k z^{-k} \right) W(z) \]

\[ w[n] = \sum_{k=1}^{N} a_k w[n-k] + x[n] \]

\[ y[n] = \sum_{k=0}^{M} b_k w[n-k] \]
6.2 Signal Flow Graph Representation of Linear Constant-Coefficient Difference Equations

Review
Example

\[ w_1[n] = x[n] + aw_2[n] + bw_2[n] \]
\[ w_2[n] = cw_1[n] \]
\[ y[n] = dx[n] + ew_2[n] \]
\[ w_1[n] = aw_4[n] + x[n] \]
\[ w_2[n] = w_1[n] \]
\[ w_3[n] = b_0 w_2[n] + b_1 w_4[n] \]
\[ w_4[n] = w_2[n-1] \]
\[ y[n] = w_3[n] \]
6.3 Basic Structure for IIR System

In this section we develop several of the most commonly used forms for implementing a linear time-invariant IIR system and obtain their flow graph representation.
6.3.1 Direct form

Difference equation:

\[ y[n] - \sum_{k=1}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k] \]

The Rational system function

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} \]
Direct form I
Direct form II
Example

Illustrate the direct form I and direct form II structures, consider the system function

\[
H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}
\]

Difference equation:

\[
y[n] = x[n] + 2x[n-1] + x[n-2] + 0.75y[n-1] - 0.125y[n-2]
\]
6.3.2 Cascade Form

\[ H(z) = A \frac{\prod_{k=1}^{M_1} (1 - g_k z^{-1}) \prod_{k=1}^{M_2} (1 - h_k z^{-1})(1 - h_k^* z^{-1})}{\prod_{k=1}^{M_1} (1 - c_k z^{-1}) \prod_{k=1}^{M_2} (1 - d_k z^{-1})(1 - d_k^* z^{-1})} \]

where \( M = M_1 + 2M_2 \) and \( N = N_1 + 2N_2 \)
It is often desirable to implement the cascade realization using a minimum of storage and computation. A modular structure is obtained by combining pairs of real factors and complex conjugate pairs

\[
H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}
\]

\(N_s\) is the largest integer contained in \((N+1)/2\)
The difference equations represented by a general cascade of direct form II second-order sections are of the form.

\[ y_0[n] = x[n] \]
\[ w_k[n] = a_{1k}w_k[n-1] + a_{2k}w_k[n-2] + y_{k-1}[n], \quad k = 1,..N_s \]
\[ y_k[n] = b_{0k}w_k[n] + b_{1k}w_k[n-1] + b_{2k}w_k[n-2], \quad k = 1,..N_s \]
\[ y[n] = y_{Ns}[n] \]
Cascade structure for a 6 order system.
Example

\[ H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})} \]
6.3.3 Parallel form

\[ H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_1} \frac{A_k}{1-c_k z^{-1}} + \sum_{k=1}^{N_2} B_k \frac{(1-e_k z^{-1})}{(1-d_k z^{-1})(1-d_k^* z^{-1})} \]

Where \( N = N_1 + 2N_2 \) if \( M \geq N \) then \( N_p = M-N \)
If coefficients $a_k$ and $b_k$ are real then $A_k, B_k, C_k, c_k$ and $e_k$ are real.

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

$N_s$ is the largest integer contained in $(N+1)/2$
\[ w_k[n] = a_{1k} w_{k}[n-1] + a_{2k} w_{k}[n-2] + x[n], \quad k = 1, .., N_s \]

\[ y_k[n] = e_{0k} w_{k}[n] + e_{1k} w_{k}[n-1], \quad k = 1, .., N_s \]

\[ y[n] = \sum_{k=0}^{N_p} C_k x[n-k] + \sum_{k=1}^{N_s} y_k[n] \]
Parallel structure for a 6 order system.
Example

Find the parallel structure for system function below

\[
H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}} = 8 + \frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}
\]

\[
= 8 + \frac{18}{1-0.5z^{-1}} - \frac{25}{1-0.25z^{-1}}
\]
6.3.4 Feedback in IIR System

*Feedback loops* - Closed paths that begin at a node and return to that node by traversing branches only in the direction of their arrowheads.

\[ y[n] = ay[n-1] + x[n] \]

\[ H(z) = \frac{1}{1 - az^{-1}} \]
6.4 Transposed Forms

Transposition of a flow graph is accomplished by reversing the directions of all branches in the network while keeping the branch transmittances as they were and reversing the roles of the input and output so that source node become sink nodes and vice versa.
Example

Find the transposed form of the first-order system.

\[ H(z) = \frac{1}{1 - az^{-1}} \]
Example

Consider the basic second-order which the difference equation:

\[ w[n] = a_1 w[n-1] + a_2 w[n-2] + x[n] \]
\[ y[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2] \]
Transpose direct form II
Transpose general direct form I
Transpose general direct form II
6.5 Basic Network Structures for FIR System

6.5.1 Direct form

For causal FIR system, the system function has only zeros (expect for poles at \( z = 0 \)), and, since the coefficients \( a_k \) are all zero, the difference equation is:

\[
y[n] = \sum_{k=0}^{M} b_k x[n - k]
\]
The impulse response

\[ h[n] = \begin{cases} 
    b_n, & n = 0, 1, \ldots, M \\
    0, & \text{otherwise}
\end{cases} \]
Direct form realization of an FIR system
Transpose form
6.5.2 Cascade Form

The cascade form for FIR systems is obtained by factoring the polynomial system function.

\[ H(z) = \sum_{k=0}^{M} h[n] z^{-n} \]

\[ = \prod_{k=1}^{M_s} \left( b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2} \right) \]

\( M_s \) is the largest integer contained in \((M+1)/2\)
Cascade form of an FIR system
6.5.3 Structures for Linear Phase FIR Systems

\[ h[M - n] = h[n] \quad \text{for } n = 0, 1, \ldots, M \]

or

\[ h[M - n] = -h[n] \quad \text{for } n = 0, 1, \ldots, M \]
\[ y[n] = \sum_{k=0}^{M} h[k]x[n-k] \]

\[ = \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=M/2+1}^{M} h[k]x[n-k] \]

\[ = \sum_{k=0}^{M/2-1} h[k]x[n-k] + h[M/2]x[n-M/2] + \sum_{k=0}^{M/2-1} h[M-k]x[n-M+k] \]
\[ h[M - n] = h[n] \quad \text{for } n = 0, 1, \ldots, M \]

\[ y[n] = \sum_{k=0}^{M/2 - 1} h[k] \{x[n-k] + x[n-M+k]\} + h[M/2]x[n-M/2] \]

\[ H(z) = \sum_{k=0}^{M/2 - 1} h[k] \left[ z^{-k} + z^{-(M-k)} \right] + h[M/2]z^{-M/2} \]
\[
\begin{align*}
    y[n] &= \sum_{k=0}^{(M-1)/2} h[k] \{x[n-k] + x[n-M+k]\} \\
    H(z) &= \sum_{k=0}^{(M-1)/2} h[k] [z^{-k} + z^{-(M-k)}] \\
    y[n] &= \sum_{k=0}^{(M-1)/2} h[k] \{x[n-k] - x[n-M+k]\} \\
    H(z) &= \sum_{k=0}^{(M-1)/2} h[k] [z^{-k} - z^{-(M-k)}]
\end{align*}
\]
\[ h[M - n] = -h[n] \quad \text{for} \ n = 0, 1, \ldots, M \]

\[ y[n] = \sum_{k=0}^{M/2-1} h[k]\{x[n-k] - x[n-M+k]\} \]

Note: \( h[M/2] = 0 \)

\[ H(z) = \sum_{k=0}^{M/2-1} h[k][z^{-k} - z^{-(M-k)}] \]
Direct form structure for an FIR linear phase system when $M$ is an even integer
Direct form structure for an FIR linear phase system when M is an odd integer
Symmetry of zeros for a linear phase FIR filter