The frequency response $H(j\omega)$ plays as important a role in the analysis of LTI systems as does its inverse transform, the unit impulse response. For one thing, since $h(t)$ completely characterizes an LTI system, then so must $H(j\omega)$. In addition, many of the properties of LTI systems can be conveniently interpreted in terms of $H(j\omega)$. For example, in Section 2.3, we saw that the impulse response of the cascade of two LTI systems is the convolution of the impulse responses of the individual systems and that the overall impulse response does not depend on the order in which the systems are cascaded. Using eq. (4.56), we can rephrase this in terms of frequency responses. As illustrated in Figure 4.19, since the impulse response of the cascade of two LTI systems is the convolution of the individual impulse responses, the convolution property then implies that the overall frequency response of the cascade of two systems is simply the product of the individual frequency responses. From this observation, it is then clear that the overall frequency response does not depend on the order of the cascade.

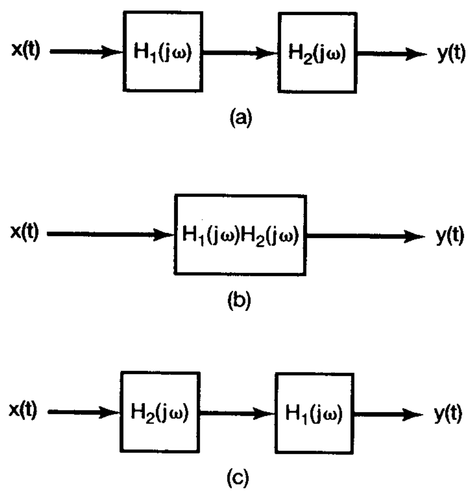

Figure 4.19 Three equivalent LTI systems. Here, each block represents an LTI system with the indicated frequency response.

As discussed in Section 4.1.2, convergence of the Fourier transform is guaranteed only under certain conditions, and consequently, the frequency response cannot be defined for every LTI system. If, however, an LTI system is stable, then, as we saw in Section 2.3.7 and Problem 2.49, its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty. \tag{4.57}$$

Equation (4.57) is one of the three Dirichlet conditions that together guarantee the existence of the Fourier transform $H(j\omega)$ of $h(t)$. Thus, assuming that $h(t)$ satisfies the other two conditions, as essentially all signals of physical or practical significance do, we see that a stable LTI system has a frequency response $H(j\omega)$.

In using Fourier analysis to study LTI systems, we will be restricting ourselves to systems whose impulse responses possess Fourier transforms. In order to use transform techniques to examine unstable LTI systems we will develop a generalization of the continuous-time Fourier transform, the Laplace transform. We defer this discussion to Chapter 9, and until then we will consider the many problems and practical applications that we can analyze using the Fourier transform.

4.4.1 Examples

To illustrate the convolution property and its applications further, let us consider several examples.

Example 4.15

Consider a continuous-time LTI system with impulse response

$$h(t) = \delta(t - t_0). \tag{4.58}$$

The frequency response of this system is the Fourier transform of $h(t)$ and is given by

$$H(j\omega) = e^{-j\omega t_0}. \tag{4.59}$$

Thus, for any input $x(t)$ with Fourier transform $X(j\omega)$, the Fourier transform of the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}X(j\omega). \end{aligned} \tag{4.60}$$

This result, in fact, is consistent with the time-shift property of Section 4.3.2. Specifically, a system for which the impulse response is $\delta(t - t_0)$ applies a time shift of t_0 to the input—that is,

$$y(t) = x(t - t_0).$$

Thus, the shifting property given in eq. (4.27) also yields eq. (4.60). Note that, either from our discussion in Section 4.3.2 or directly from eq. (4.59), the frequency response of a system that is a pure time shift has unity magnitude at all frequencies (i.e., $|e^{-j\omega t_0}| = 1$) and has a phase characteristic $-\omega t_0$ that is a linear function of ω.

Example 4.16

As a second example, let us examine a differentiator—that is, an LTI system for which the input $x(t)$ and the output $y(t)$ are related by

$$y(t) = \frac{dx(t)}{dt}.$$

From the differentiation property of Section 4.3.4,

$$Y(j\omega) = j\omega X(j\omega). \tag{4.61}$$

Consequently, from eq. (4.56), it follows that the frequency response of a differentiator is

$$H(j\omega) = j\omega. \tag{4.62}$$

Example 4.17

Consider an integrator—that is, an LTI system specified by the equation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

The impulse response for this system is the unit step $u(t)$, and therefore, from Example 4.11 and eq. (4.33), the frequency response of the system is

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Then using eq. (4.56), we have

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(j\omega)\delta(\omega) \\ &= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega), \end{aligned}$$

which is consistent with the integration property of eq. (4.32).

Example 4.18

As we discussed in Section 3.9.2, frequency-selective filtering is accomplished with an LTI system whose frequency response $H(j\omega)$ passes the desired range of frequencies and significantly attenuates frequencies outside that range. For example, consider the ideal lowpass filter introduced in Section 3.9.2, which has the frequency reponse illustrated in Figure 4.20 and given by

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}. \tag{4.63}$$

Now that we have developed the Fourier transform representation, we know that the impulse response $h(t)$ of this ideal filter is the inverse transform of eq. (4.63). Using the result in Example 4.5, we then have

$$h(t) = \frac{\sin \omega_c t}{\pi t}, \tag{4.64}$$

which is plotted in Figure 4.21.

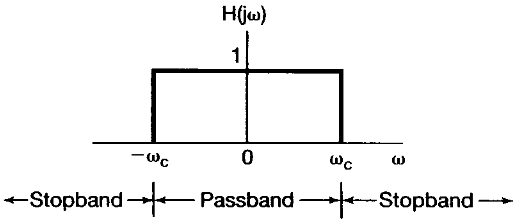

Figure 4.20 Frequency response of an ideal lowpass filter.

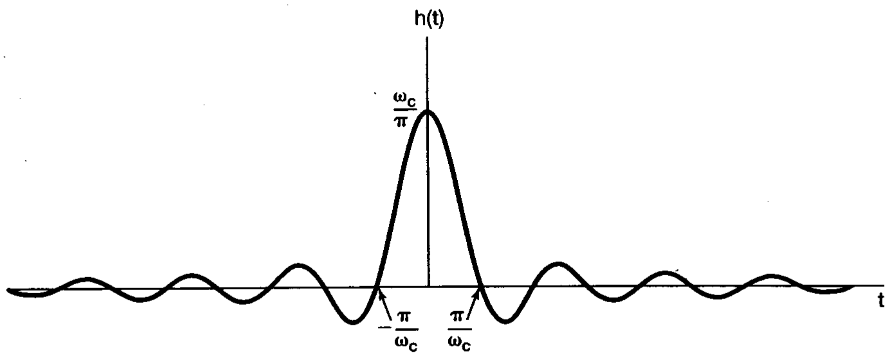

Figure 4.21 Impulse response of an ideal lowpass filter.

From Example 4.18, we can begin to see some of the issues that arise in filter design that involve looking in both the time and frequency domains. In particular, while the ideal lowpass filter does have perfect frequency selectivity, its impulse response has some characteristics that may not be desirable. First, note that $h(t)$ is not zero for $t < 0$. Consequently, the ideal lowpass filter is not causal, and thus, in applications requiring causal systems, the ideal filter is not an option. Moreover, as we discuss in Chapter 6, even if causality is not an essential constraint, the ideal filter is not easy to approximate closely, and non-ideal filters that are more easily implemented are typically preferred. Furthermore, in some applications (such as the automobile suspension system discussed in Section 6.7.1), oscillatory behavior in the impulse response of a lowpass filter may be undesirable. In such applications the time domain characteristics of the ideal lowpass filter, as shown in Figure 4.21, may be unacceptable, implying that we may need to trade off frequency-domain characteristics such as ideal frequency selectivity with time-domain properties.

For example, consider the LTI system with impulse response

$$h(t) = e^{-t}u(t). \tag{4.65}$$

The frequency response of this system is

$$H(j\omega) = \frac{1}{j\omega + 1}. \tag{4.66}$$

Comparing eqs. (3.145) and (4.66), we see that this system can be implemented with the simple RC circuit discussed in Section 3.10. The impulse response and the magnitude of the frequency response are shown in Figure 4.22. While the system does not have the strong frequency selectivity of the ideal lowpass filter, it is causal and has an impulse response that decays monotonically, i.e., without oscillations. This filter or somewhat more complex ones corresponding to higher order differential equations are quite frequently preferred to ideal filters because of their causality, ease of implementation, and flexibility in allowing trade-offs, among other design considerations such as frequency selectivity and oscillatory behavior in the time domain. Many of these issues will be discussed in more detail in Chapter 6.

The convolution property is often useful in evaluating the convolution integral—i.e., in computing the response of LTI systems. This is illustrated in the next example.

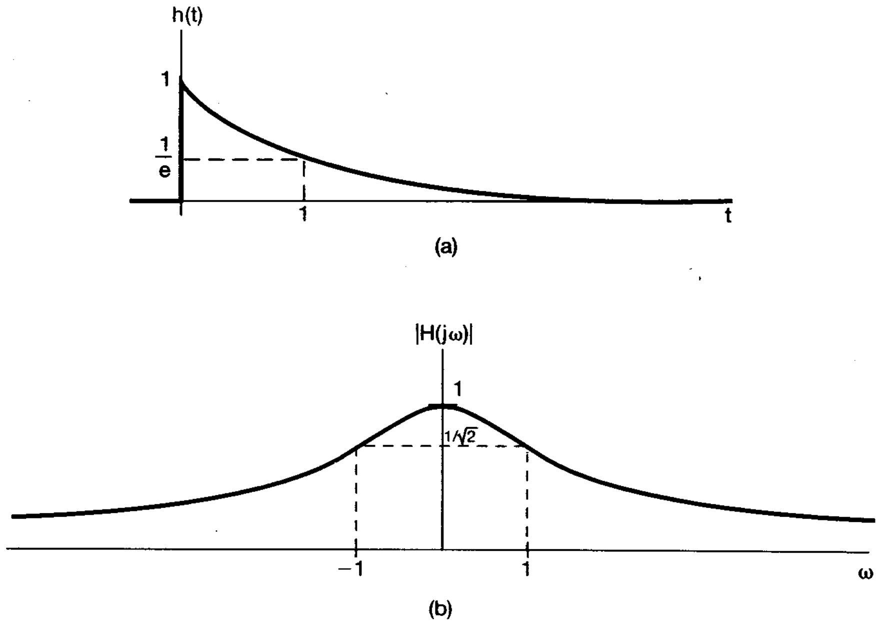

Figure 4.22 (a) Impulse response of the LTI system in eq. (4.65); (b) magnitude of the frequency response of the system.

Example 4.19

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \quad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \quad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, let us transform the problem into the frequency domain. From Example 4.1, the Fourier transforms of $x(t)$ and $h(t)$ are

$$X(j\omega) = \frac{1}{b + j\omega}$$

and

$$H(j\omega) = \frac{1}{a + j\omega}.$$

Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}. \tag{4.67}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion. Such expansions are extremely useful in evaluating inverse transforms, and the general method for performing a partial-fraction expansion is developed in the appendix. For this example, assuming that $b \neq a$, the partial fraction expansion for $Y(j\omega)$ takes the form

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}, \tag{4.68}$$

where A and B are constants to be determined. One way to find A and B is to equate the right-hand sides of eqs. (4.67) and (4.68), multiply both sides by $(a + j\omega)(b + j\omega)$, and solve for A and B. Alternatively, in the appendix we present a more general and efficient method for computing the coefficients in partial-fraction expansions such as eq. (4.68). Using either of these approaches, we find that

$$A = \frac{1}{b - a} = -B,$$

and therefore,

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]. \tag{4.69}$$

The inverse transform for each of the two terms in eq. (4.69) can be recognized by inspection. Using the linearity property of Section 4.3.1, we have

$$y(t) = \frac{1}{b - a}[e^{-at}u(t) - e^{-bt}u(t)].$$

When $b = a$, the partial fraction expansion of eq. (4.69) is not valid. However, with $b = a$, eq. (4.67) becomes

$$Y(j\omega) = \frac{1}{(a + j\omega)^2}.$$

Recognizing this as

$$\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right],$$

we can use the dual of the differentiation property, as given in eq. (4.40). Thus,

$$e^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2},$$

and consequently,

$$y(t) = te^{-at}u(t).$$

Example 4.20

As another illustration of the usefulness of the convolution property, let us consider the problem of determining the response of an ideal lowpass filter to an input signal $x(t)$ that has the form of a sinc function. That is,

$$x(t) = \frac{\sin \omega_i t}{\pi t}.$$

Of course, the impulse response of the ideal lowpass filter is of a similar form, namely,

$$h(t) = \frac{\sin \omega_c t}{\pi t}.$$

The filter output $y(t)$ will therefore be the convolution of two sinc functions, which, as we now show, also turns out to be a sinc function. A particularly convenient way of deriving this result is to first observe that

$$Y(j\omega) = X(j\omega)H(j\omega),$$

where

$$X(j\omega) = \begin{cases} 1 & |\omega| \le \omega_i \\ 0 & \text{elsewhere} \end{cases}$$

and

$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \text{elsewhere} \end{cases}.$$

Therefore,

$$Y(j\omega) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & \text{elsewhere} \end{cases},$$

where ω_0 is the smaller of the two numbers ω_i and ω_c. Finally, the inverse Fourier transform of $Y(j\omega)$ is given by

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} & \text{if } \omega_c \le \omega_i \\ \dfrac{\sin \omega_i t}{\pi t} & \text{if } \omega_i \le \omega_c \end{cases}$$

That is, depending upon which of ω_c and ω_i is smaller, the output is equal to either $x(t)$ or $h(t)$.

4.5 THE MULTIPLICATION PROPERTY

The convolution property states that convolution in the *time* domain corresponds to multiplication in the *frequency* domain. Because of duality between the time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$\boxed{r(t) = s(t)p(t) \longleftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta} \qquad (4.70)$$

This can be shown by exploiting duality as discussed in Section 4.3.6, together with the convolution property, or by directly using the Fourier transform relations in a manner analogous to the procedure used in deriving the convolution property.

Multiplication of one signal by another can be thought of as using one signal to scale or *modulate* the amplitude of the other, and consequently, the multiplication of two signals is often referred to as *amplitude modulation*. For this reason, eq. (4.70) is sometimes referred to as the *modulation property*. As we shall see in Chapters 7 and 8, this property has several very important applications. To illustrate eq. (4.70), and to suggest one of the applications that we will discuss in subsequent chapters, let us consider several examples.

Example 4.21

Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is depicted in Figure 4.23(a). Also, consider the signal

$$p(t) = \cos \omega_0 t.$$

Then

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0),$$

as sketched in Figure 4.23(b), and the spectrum $R(j\omega)$ of $r(t) = s(t)p(t)$ is obtained by

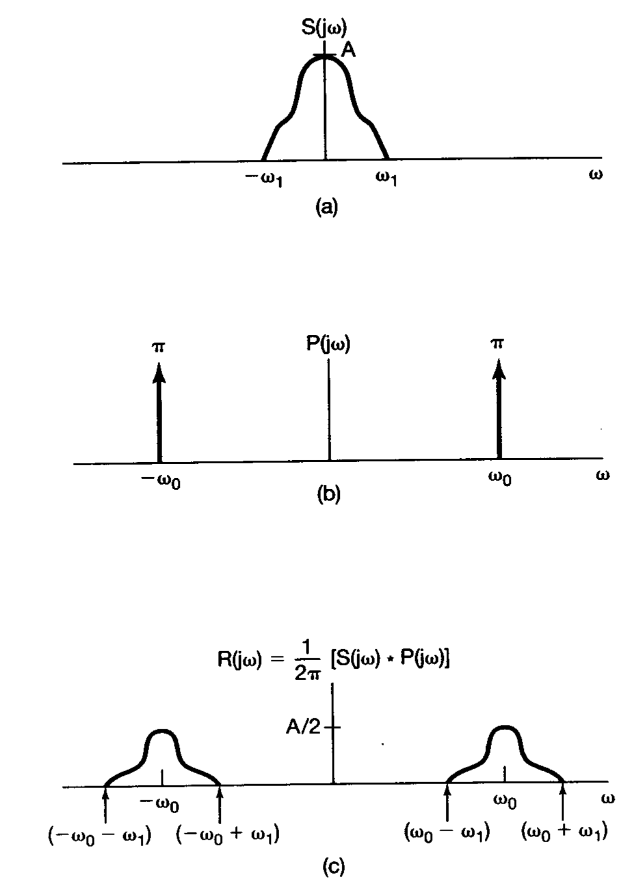

Figure 4.23 Use of the multiplication property in Example 4.21: (a) the Fourier transform of a signal $s(t)$; (b) the Fourier transform of $p(t) = \cos \omega_0 t$; (c) the Fourier transform of $r(t) = s(t)p(t)$.

an application of eq. (4.70), yielding

$$\begin{aligned} R(j\omega) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta \\ &= \frac{1}{2}S(j(\omega-\omega_0)) + \frac{1}{2}S(j(\omega+\omega_0)), \end{aligned} \tag{4.71}$$

which is sketched in Figure 4.23(c). Here we have assumed that $\omega_0 > \omega_1$, so that the two nonzero portions of $R(j\omega)$ do not overlap. Clearly, the spectrum of $r(t)$ consists of the sum of two shifted and scaled versions of $S(j\omega)$.

From eq. (4.71) and from Figure 4.23, we see that all of the information in the signal $s(t)$ is preserved when we multiply this signal by a sinusoidal signal, although the information has been shifted to higher frequencies. This fact forms the basis for sinusoidal amplitude modulation systems for communications. In the next example, we learn how we can recover the original signal $s(t)$ from the amplitude-modulated signal $r(t)$.

Example 4.22

Let us now consider $r(t)$ as obtained in Example 4.21, and let

$$g(t) = r(t)p(t),$$

where, again, $p(t) = \cos\omega_0 t$. Then, $R(j\omega)$, $P(j\omega)$, and $G(j\omega)$ are as shown in Figure 4.24.

From Figure 4.24(c) and the linearity of the Fourier transform, we see that $g(t)$ is the sum of $(1/2)s(t)$ and a signal with a spectrum that is nonzero only at higher frequen-

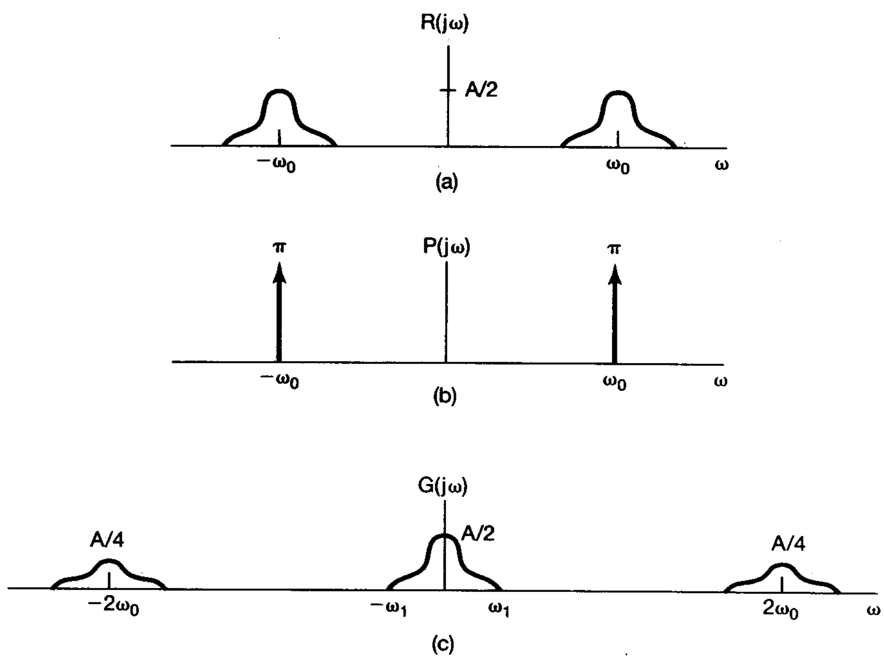

Figure 4.24 Spectra of signals considered in Example 4.22: (a) $R(j\omega)$; (b) $P(j\omega)$; (c) $G(j\omega)$.

cies (centered around $\pm 2\omega_0$). Suppose then that we apply the signal $g(t)$ as the input to a frequency-selective lowpass filter with frequency response $H(j\omega)$ that is constant at low frequencies (say, for $|\omega| < \omega_1$) and zero at high frequencies (for $|\omega| > \omega_1$). Then the output of this system will have as its spectrum $H(j\omega)G(j\omega)$, which, because of the particular choice of $H(j\omega)$, will be a scaled replica of $S(j\omega)$. Therefore, the output itself will be a scaled version of $s(t)$. In Chapter 8, we expand significantly on this idea as we develop in detail the fundamentals of amplitude modulation.

Example 4.23

Another illustration of the usefulness of the Fourier transform multiplication property is provided by the problem of determining the Fourier transform of the signal

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}.$$

The key here is to recognize $x(t)$ as the product of two sinc functions:

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right).$$

Applying the multiplication property of the Fourier transform, we obtain

$$X(j\omega) = \frac{1}{2}\mathfrak{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathfrak{F}\left\{\frac{\sin(t/2)}{\pi t}\right\}.$$

Noting that the Fourier transform of each sinc function is a rectangular pulse, we can proceed to convolve those pulses to obtain the function $X(j\omega)$ displayed in Figure 4.25.

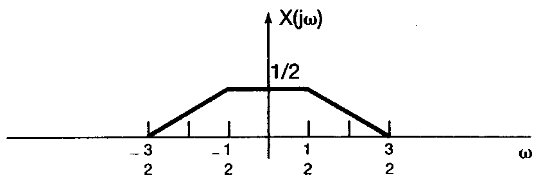

Figure 4.25 The Fourier transform of $x(t)$ in Example 4.23.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

As suggested in Examples 4.21 and 4.22 and developed more fully in Chapter 8, one of the important applications of the multiplication property is amplitude modulation in communication systems. Another important application is in the implementation of frequency-selective bandpass filters with tunable center frequencies that can be adjusted by the simple turn of a dial. In a frequency-selective bandpass filter built with elements such as resistors, operational amplifiers, and capacitors, the center frequency depends on a number of element values, all of which must be varied simultaneously in the correct way if the center frequency is to be adjusted directly. This is generally difficult and cumbersome in comparison with building a filter whose characteristics are fixed. An alternative to directly varying the filter characteristics is to use a fixed frequency-selective filter and shift the spectrum of the signal appropriately, using the principles of sinusoidal amplitude modulation.

For example, consider the system shown in Figure 4.26. Here, an input signal $x(t)$ is multiplied by the complex exponential signal $e^{j\omega_c t}$. The resulting signal is then passed through a lowpass filter with cutoff frequency ω_0, and the output is multiplied by $e^{-j\omega_c t}$. The spectra of the signals $x(t)$, $y(t)$, $w(t)$, and $f(t)$ are illustrated in Figure 4.27.

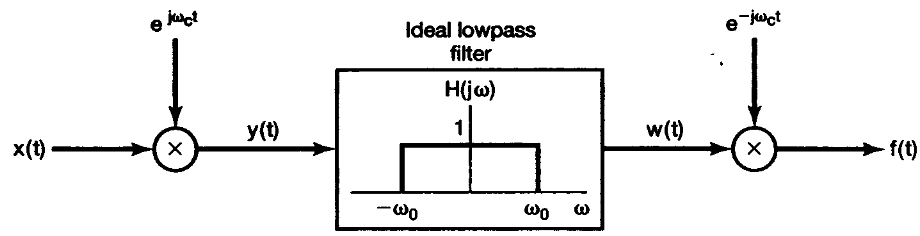

Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

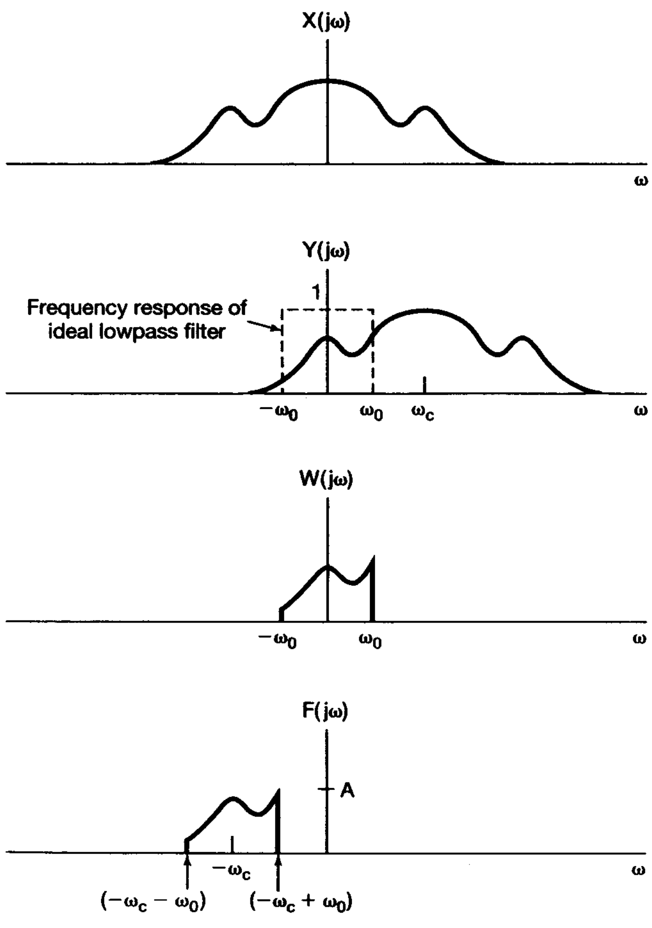

Figure 4.27 Spectra of the signals in the system of Figure 4.26.

Specifically, from either the multiplication property or the frequency-shifting property it follows that the Fourier transform of $y(t) = e^{j\omega_c t}x(t)$ is

$$Y(j\omega) = \int_{-\infty}^{+\infty} \delta(\theta - \omega_c)X(\omega - \theta)d\theta$$

so that $Y(j\omega)$ equals $X(j\omega)$ shifted to the right by ω_c and frequencies in $X(j\omega)$ near $\omega = \omega_c$ have been shifted into the passband of the lowpass filter. Similarly, the Fourier transform of $f(t) = e^{-j\omega_c t}w(t)$ is

$$F(j\omega) = W(j(\omega + \omega_c)),$$

so that the Fourier transform of $F(j\omega)$ is $W(j\omega)$ shifted to the left by ω_c. From Figure 4.27, we observe that the overall system of Figure 4.26 is equivalent to an ideal bandpass filter with center frequency $-\omega_c$ and bandwidth $2\omega_0$, as illustrated in Figure 4.28. As the frequency ω_c of the complex exponential oscillator is varied, the center frequency of the bandpass filter varies.

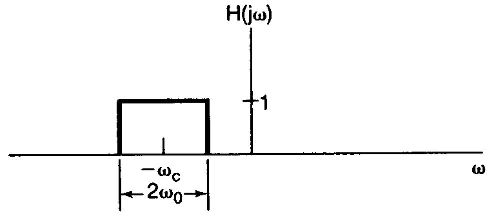

Figure 4.28 Bandpass filter equivalent of Figure 4.26.

In the system of Figure 4.26 with $x(t)$ real, the signals $y(t)$, $w(t)$, and $f(t)$ are all complex. If we retain only the real part of $f(t)$, the resulting spectrum is that shown in Figure 4.29, and the equivalent bandpass filter passes bands of frequencies centered around ω_c and $-\omega_c$, as indicated in Figure 4.30. Under certain conditions, it is also possible to use sinusoidal rather than complex exponential modulation to implement the system of the latter figure. This is explored further in Problem 4.46.

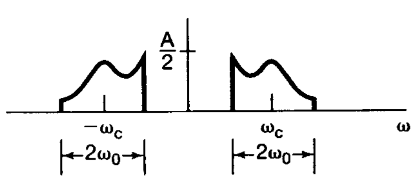

Figure 4.29 Spectrum of $\mathcal{R}e\{f(t)\}$ associated with Figure 4.26.

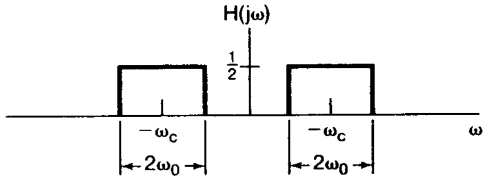

Figure 4.30 Equivalent bandpass filter for $\mathcal{R}e\{f(t)\}$ in Figure 4.29.

4.6 TABLES OF FOURIER PROPERTIES AND OF BASIC FOURIER TRANSFORM PAIRS

In the preceding sections and in the problems at the end of the chapter, we have considered some of the important properties of the Fourier transform. These are summarized in Table 4.1, in which we have also indicated the section of this chapter in which each property has been discussed.

In Table 4.2, we have assembled a list of many of the basic and important Fourier transform pairs. We will encounter many of these repeatedly as we apply the tools of

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$
4.3.7	Parseval's Relation for Aperiodic Signals $\int_{-\infty}^{+\infty} \|x(t)\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|X(j\omega)\|^2 d\omega$		

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \; k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & T_1 < \lvert t \rvert \le \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t \rvert < T_1 \\ 0, & \lvert t \rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$, $\Re e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Fourier analysis in our examination of signals and systems. All of the transform pairs, except for the last one in the table, have been considered in examples in the preceding sections. The last pair is considered in Problem 4.40. In addition, note that several of the signals in Table 4.2 are periodic, and for these we have also listed the corresponding Fourier series coefficients.

4.7 SYSTEMS CHARACTERIZED BY LINEAR CONSTANT-COEFFICIENT DIFFERENTIAL EQUATIONS

As we have discussed on several occasions, a particularly important and useful class of continuous-time LTI systems is those for which the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \tag{4.72}$$

In this section, we consider the question of determining the frequency response of such an LTI system. Throughout the discussion we will always assume that the frequency response of the system exists, i.e., that eq. (3.121) converges.

There are two closely related ways in which to determine the frequency response $H(j\omega)$ for an LTI system described by the differential equation (4.72). The first of these, which relies on the fact that complex exponential signals are eigenfunctions of LTI systems, was used in Section 3.10 in our analysis of several simple, nonideal filters. Specifically, if $x(t) = e^{j\omega t}$, then the output must be $y(t) = H(j\omega)e^{j\omega t}$. Substituting these expressions into the differential equation (4.72) and performing some algebra, we can then solve for $H(j\omega)$. In this section we use an alternative approach to arrive at the same answer, making use of the differentiation property, eq. (4.31), of Fourier transforms.

Consider an LTI system characterized by eq. (4.72). From the convolution property,

$$Y(j\omega) = H(j\omega)X(j\omega),$$

or equivalently,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \tag{4.73}$$

where $X(j\omega)$, $Y(j\omega)$, and $H(j\omega)$ are the Fourier transforms of the input $x(t)$, output $y(t)$, and impulse response $h(t)$, respectively. Next, consider applying the Fourier transform to both sides of eq. (4.72) to obtain

$$\mathfrak{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathfrak{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}. \tag{4.74}$$

From the linearity property, eq. (4.26), this becomes

$$\sum_{k=0}^{N} a_k \mathfrak{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathfrak{F}\left\{\frac{d^k x(t)}{dt^k}\right\}, \tag{4.75}$$

and from the differentiation property, eq. (4.31),

$$\sum_{k=0}^{N} a_k(j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k(j\omega)^k X(j\omega),$$

or equivalently,

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k(j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k(j\omega)^k\right].$$

Thus, from eq. (4.73),

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k(j\omega)^k}{\sum_{k=0}^{N} a_k(j\omega)^k}. \tag{4.76}$$

Observe that $H(j\omega)$ is thus a rational function; that is, it is a ratio of polynomials in $(j\omega)$. The coefficients of the numerator polynomial are the same coefficients as those that appear on the right-hand side of eq. (4.72), and the coefficients of the denominator polynomial are the same coefficients as appear on the left side of eq. (4.72). Hence, the frequency response given in eq. (4.76) for the LTI system characterized by eq. (4.72) can be written down directly by inspection.

The differential equation (4.72) is commonly referred to as an Nth-order differential equation, as the equation involves derivatives of the output $y(t)$ up through the Nth derivative. Also, the denominator of $H(j\omega)$ in eq. (4.76) is an Nth-order polynomial in $(j\omega)$.

Example 4.24

Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{4.77}$$

with $a > 0$. From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}. \tag{4.78}$$

Comparing this with the result of Example 4.1, we see that eq. (4.78) is the Fourier transform of $e^{-at}u(t)$. The impulse response of the system is then recognized as

$$h(t) = e^{-at}u(t).$$

Example 4.25

Consider a stable LTI system that is characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{(j\omega)+2}{(j\omega)^2 + 4(j\omega) + 3}. \tag{4.79}$$

To determine the corresponding impulse response, we require the inverse Fourier transform of $H(j\omega)$. This can be found using the technique of partial-fraction expansion employed in Example 4.19 and discussed in detail in the appendix. (In particular, see Example A.1, in which the details of the calculations for the partial-fraction expansion of eq. (4.79) are worked out.) As a first step, we factor the denominator of the right-hand side of eq. (4.79) into a product of lower order terms:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}. \tag{4.80}$$

Then, using the method of partial-fraction expansion, we find that

$$H(j\omega) = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}.$$

The inverse transform of each term can be recognized from Example 4.24, with the result that

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

The procedure used in Example 4.25 to obtain the inverse Fourier transform is generally useful in inverting transforms that are ratios of polynomials in $j\omega$. In particular, we can use eq. (4.76) to determine the frequency response of any LTI system described by a linear constant-coefficient differential equation and then can calculate the impulse response by performing a partial-fraction expansion that puts the frequency response into a form in which the inverse transform of each term can be recognized by inspection. In addition, if the Fourier transform $X(j\omega)$ of the input to such a system is also a ratio of polynomials in $j\omega$, then so is $Y(j\omega) = H(j\omega)X(j\omega)$. In this case we can use the same technique to solve the differential equation—that is, to find the response $y(t)$ to the input $x(t)$. This is illustrated in the next example.

Example 4.26

Consider the system of Example 4.25, and suppose that the input is

$$x(t) = e^{-t}u(t).$$

Then, using eq. (4.80), we have

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}\right]\left[\frac{1}{j\omega + 1}\right]$$

$$= \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}. \tag{4.81}$$

As discussed in the appendix, in this case the partial-fraction expansion takes the form

$$Y(j\omega) = \frac{A_{11}}{j\omega + 1} + \frac{A_{12}}{(j\omega + 1)^2} + \frac{A_{21}}{j\omega + 3}, \tag{4.82}$$

where A_{11}, A_{12}, and A_{21} are constants to be determined. In Example A.2 in the appendix, the technique of partial-fraction expansion is used to determine these constants. The values obtained are

$$A_{11} = \frac{1}{4}, \quad A_{12} = \frac{1}{2}, \quad A_{21} = -\frac{1}{4},$$

so that

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}. \tag{4.83}$$

Again, the inverse Fourier transform for each term in eq. (4.83) can be obtained by inspection. The first and third terms are of the same type that we have encountered in the preceding two examples, while the inverse transform of the second term can be obtained from Table 4.2 or, as was done in Example 4.19, by applying the dual of the differentiation property, as given in eq. (4.40), to $1/(j\omega + 1)$. The inverse transform of eq. (4.83) is then found to be

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

From the preceding examples, we see how the techniques of Fourier analysis allow us to reduce problems concerning LTI systems characterized by differential equations to straightforward algebraic problems. This important fact is illustrated further in a number of the problems at the end of the chapter. In addition (see Chapter 6), the algebraic structure of the rational transforms encountered in dealing with LTI systems described by differential equations greatly facilitate the analysis of their frequency-domain properties and the development of insights into both the time-domain and frequency-domain characteristics of this important class of systems.

4.8 SUMMARY

In this chapter, we have developed the Fourier transform representation for continous-time signals and have examined many of the properties that make this transform so useful. In particular, by viewing an aperiodic signal as the limit of a periodic signal as the period becomes arbitrarily large, we derived the Fourier transform representation for aperiodic signals from the Fourier series representation for periodic signals developed in Chapter 3. In addition, periodic signals themselves can be represented using Fourier transforms consisting of trains of impulses located at the harmonic frequencies of the periodic signal and with areas proportional to the corresponding Fourier series coefficients.

The Fourier transform possesses a wide variety of important properties that describe how different characteristics of signals are reflected in their transforms, and in this chapter we have derived and examined many of these properties. Among them are two that have particular significance for our study of signals and systems. The first is the convolution property, which is a direct consequence of the eigenfunction property of complex exponential signals and which leads to the description of an LTI system in terms of its frequency response. This description plays a fundamental role in the frequency-domain approach to the analysis of LTI systems, which we will continue to explore in subsequent chapters. The second property of the Fourier transform that has extremely important implications is the multiplication property, which provides the basis for the frequency-domain analysis of sampling and modulation systems. We examine these systems further in Chapters 7 and 8.

We have also seen that the tools of Fourier analysis are particularly well suited to the examination of LTI systems characterized by linear constant-coefficient differential equations. Specifically, we have found that the frequency response for such a system can be determined by inspection and that the technique of partial-fraction expansion can then be used to facilitate the calculation of the impulse response of the system. In subsequent chapters, we will find that the convenient algebraic structure of the frequency responses of these systems allows us to gain considerable insight into their characteristics in both the time and frequency domains.

Chapter 4 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

4.1. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $e^{-2(t-1)}u(t-1)$ **(b)** $e^{-2|t-1|}$

Sketch and label the magnitude of each Fourier transform.

4.2. Use the Fourier transform analysis equation (4.9) to calculate the Fourier transforms of:

(a) $\delta(t+1)+\delta(t-1)$ **(b)** $\frac{d}{dt}\{u(-2-t)+u(t-2)\}$

Sketch and label the magnitude of each Fourier transform.

4.3. Determine the Fourier transform of each of the following periodic signals:

(a) $\sin(2\pi t+\frac{\pi}{4})$ **(b)** $1+\cos(6\pi t+\frac{\pi}{8})$

4.4. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transforms of:

(a) $X_1(j\omega) = 2\pi\,\delta(\omega)+\pi\,\delta(\omega-4\pi)+\pi\,\delta(\omega+4\pi)$

(b) $X_2(j\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega < 0 \\ 0, & |\omega| > 2 \end{cases}$

4.5. Use the Fourier transform synthesis equation (4.8) to determine the inverse Fourier transform of $X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$, where

$$|X(j\omega)| = 2\{u(\omega + 3) - u(\omega - 3)\},$$

$$\sphericalangle X(j\omega) = -\frac{3}{2}\omega + \pi.$$

Use your answer to determine the values of t for which $x(t) = 0$.

4.6. Given that $x(t)$ has the Fourier transform $X(j\omega)$, express the Fourier transforms of the signals listed below in terms of $X(j\omega)$. You may find useful the Fourier transform properties listed in Table 4.1.

(a) $x_1(t) = x(1 - t) + x(-1 - t)$
(b) $x_2(t) = x(3t - 6)$
(c) $x_3(t) = \frac{d^2}{dt^2}x(t - 1)$

4.7. For each of the following Fourier transforms, use Fourier transform properties (Table 4.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X_1(j\omega) = u(\omega) - u(\omega - 2)$
(b) $X_2(j\omega) = \cos(2\omega)\sin(\frac{\omega}{2})$
(c) $X_3(j\omega) = A(\omega)e^{jB(\omega)}$, where $A(\omega) = (\sin 2\omega)/\omega$ and $B(\omega) = 2\omega + \frac{\pi}{2}$
(d) $X(j\omega) = \sum_{k=-\infty}^{\infty}(\frac{1}{2})^{|k|}\,\delta(\omega - \frac{k\pi}{4})$

4.8. Consider the signal

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2}. \\ 1, & t > \frac{1}{2} \end{cases}$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.
(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

4.9. Consider the signal

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t + 1)/2, & -1 \le t \le 1 \end{cases}.$$

(a) With the help of Tables 4.1 and 4.2, determine the closed-form expression for $X(j\omega)$.
(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$.
(c) What is the Fourier transform of the odd part of $x(t)$?

4.10. **(a)** Use Tables 4.1 and 4.2 to help determine the Fourier transform of the following signal:

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

(b) Use Parseval's relation and the result of the previous part to determine the numerical value of

$$A = \int_{-\infty}^{+\infty} t^2\left(\frac{\sin t}{\pi t}\right)^4 dt$$

4.11. Given the relationships

$$y(t) = x(t) * h(t)$$

and

$$g(t) = x(3t) * h(3t),$$

and given that $x(t)$ has Fourier transform $X(j\omega)$ and $h(t)$ has Fourier transform $H(j\omega)$, use Fourier transform properties to show that $g(t)$ has the form

$$g(t) = Ay(Bt).$$

Determine the values of A and B.

4.12. Consider the Fourier transform pair

$$e^{-|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

(a) Use the appropriate Fourier transform properties to find the Fourier transform of $te^{-|t|}$.

(b) Use the result from part (a), along with the duality property, to determine the Fourier transform of

$$\frac{4t}{(1+t^2)^2}.$$

Hint: See Example 4.13.

4.13. Let $x(t)$ be a signal whose Fourier transform is

$$X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5),$$

and let

$$h(t) = u(t) - u(t-2).$$

(a) Is $x(t)$ periodic?

(b) Is $x(t) * h(t)$ periodic?

(c) Can the convolution of two aperiodic signals be periodic?

4.14. Consider a signal $x(t)$ with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real and nonnegative.
2. $\mathfrak{F}^{-1}\{(1 + j\omega)X(j\omega)\} = Ae^{-2t}u(t)$, where A is independent of t.
3. $\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = 2\pi$.

Determine a closed-form expression for $x(t)$.

4.15. Let $x(t)$ be a signal with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

1. $x(t)$ is real.
2. $x(t) = 0$ for $t \le 0$.
3. $\frac{1}{2\pi}\int_{-\infty}^{\infty} \Re e\{X(j\omega)\}e^{j\omega t} \, d\omega = |t|e^{-|t|}$.

Determine a closed-form expression for $x(t)$.

4.16. Consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{4})}{(k\frac{\pi}{4})} \delta(t - k\frac{\pi}{4}).$$

(a) Determine $g(t)$ such that

$$x(t) = \left(\frac{\sin t}{\pi t}\right) g(t).$$

(b) Use the multiplication property of the Fourier transform to argue that $X(j\omega)$ is periodic. Specify $X(j\omega)$ over one period.

4.17. Determine whether each of the following statements is true or false. Justify your answers.

(a) An odd and imaginary signal always has an odd and imaginary Fourier transform.

(b) The convolution of an odd Fourier transform with an even Fourier transform is always odd.

4.18. Find the impulse response of a system with the frequency response

$$H(j\omega) = \frac{(\sin^2(3\omega))\cos\omega}{\omega^2}.$$

4.19. Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}.$$

For a particular input $x(t)$ this system is observed to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

Determine $x(t)$.

4.20. Find the impulse response of the causal LTI system represented by the *RLC* circuit considered in Problem 3.20. Do this by taking the inverse Fourier transform of the circuit's frequency response. You may use Tables 4.1 and 4.2 to help evaluate the inverse Fourier transform.

BASIC PROBLEMS

4.21. Compute the Fourier transform of each of the following signals:

(a) $[e^{-\alpha t}\cos\omega_0 t]u(t),\ \alpha > 0$

(b) $\epsilon^{-3|t|}\sin 2t$

(c) $x(t) = \begin{cases} 1+\cos\pi t, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

(d) $\sum_{k=0}^{\infty}\alpha^k\,\delta(t-kT),\ |\alpha|<1$

(e) $[te^{-2t}\sin 4t]u(t)$

(f) $\left[\frac{\sin\pi t}{\pi t}\right]\left[\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right]$

(g) $x(t)$ as shown in Figure P4.21(a)

(h) $x(t)$ as shown in Figure P4.21(b)

(i) $x(t) = \begin{cases} 1-t^2, & 0<t<1 \\ 0, & \text{otherwise} \end{cases}$

(j) $\sum_{n=-\infty}^{+\infty} e^{-|t-2n|}$

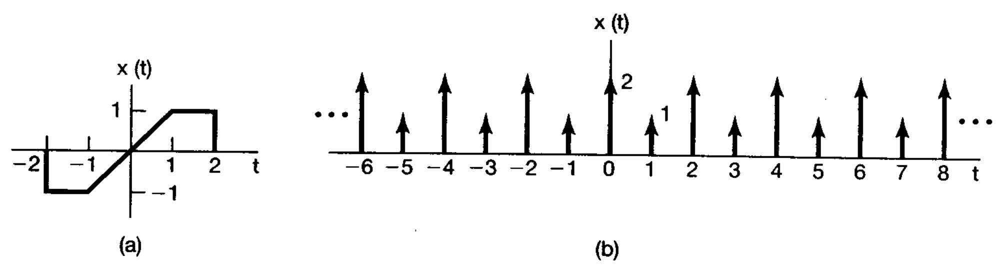

Figure P4.21

4.22. Determine the continuous-time signal corresponding to each of the following transforms.

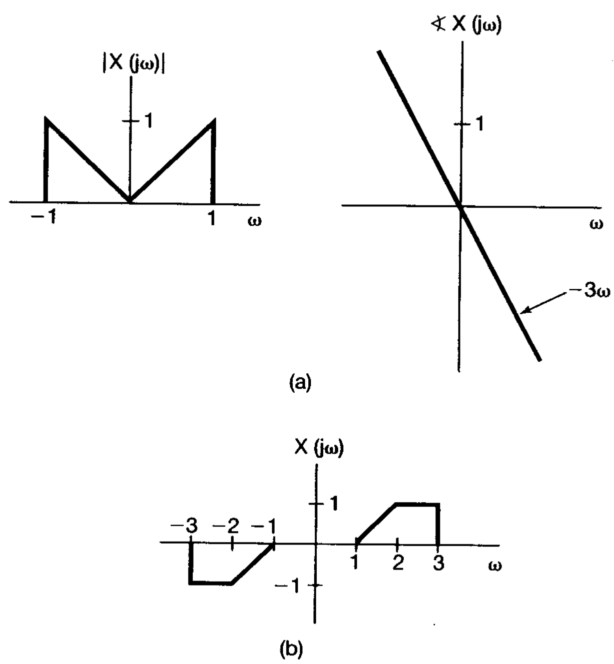

Figure P4.22

(a) $X(j\omega) = \frac{2\sin[3(\omega-2\pi)]}{(\omega-2\pi)}$
(b) $X(j\omega) = \cos(4\omega + \pi/3)$
(c) $X(j\omega)$ as given by the magnitude and phase plots of Figure P4.22(a)
(d) $X(j\omega) = 2[\delta(\omega - 1) - \delta(\omega + 1)] + 3[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)]$
(e) $X(j\omega)$ as in Figure P4.22(b)

4.23. Consider the signal

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.

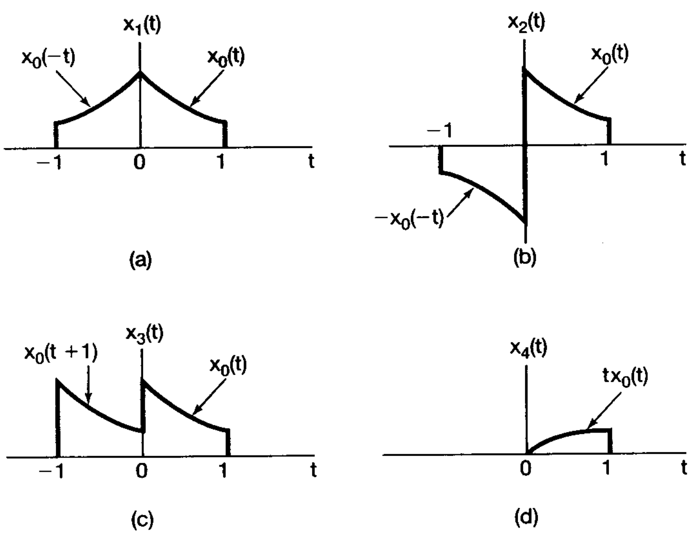

Figure P4.23

4.24. **(a)** Determine which, if any, of the real signals depicted in Figure P4.24 have Fourier transforms that satisfy each of the following conditions:
(1) $\Re e\{X(j\omega)\} = 0$
(2) $\Im m\{X(j\omega)\} = 0$
(3) There exists a real α such that $e^{j\alpha\omega}X(j\omega)$ is real
(4) $\int_{-\infty}^{\infty} X(j\omega)\,d\omega = 0$
(5) $\int_{-\infty}^{\infty} \omega X(j\omega)\,d\omega = 0$
(6) $X(j\omega)$ is periodic

(b) Construct a signal that has properties (1), (4), and (5) and does *not* have the others.

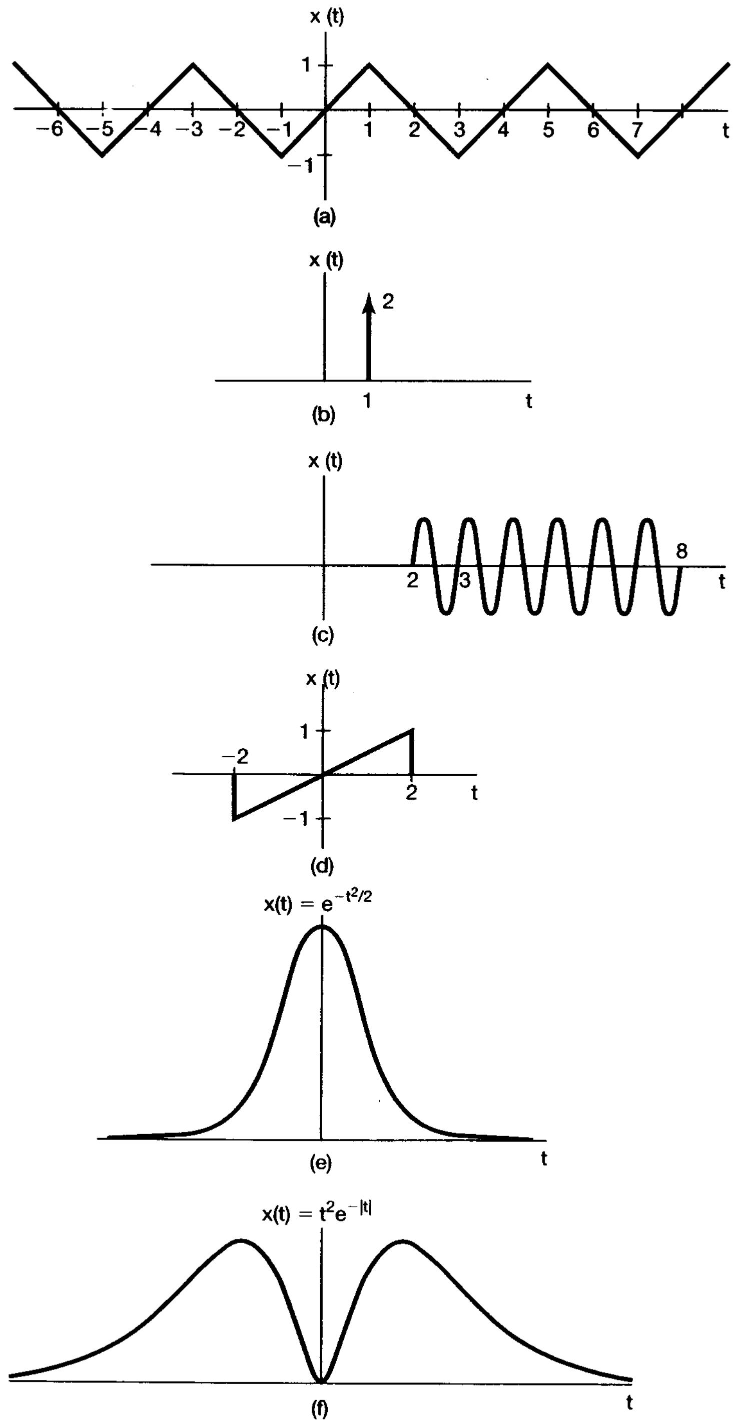
x (t)
1
−1
−6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
t
(a)
x (t)
2
1
t
(b)
x (t)
2
3
8
t
(c)
x (t)
1
−2
2
−1
t
(d)
x(t) = e^{−t²/2}
t
(e)
x(t) = t²e^{−|t|}
t
(f)

Figure P4.24

4.25. Let $X(j\omega)$ denote the Fourier transform of the signal $x(t)$ depicted in Figure P4.25.

(a) $X(j\omega)$ can be expressed as $A(j\omega)e^{j\Theta(j\omega)}$, where $A(j\omega)$ and $\Theta(j\omega)$ are both real-values. Find $\Theta(j\omega)$.

(b) Find $X(j0)$.

(c) Find $\int_{-\infty}^{\infty} X(j\omega)\, d\omega$.

(d) Evaluate $\int_{-\infty}^{\infty} X(j\omega)\frac{2\sin\omega}{\omega}e^{j2\omega}\, d\omega$.

(e) Evaluate $\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$.

(f) Sketch the inverse Fourier transform of $\mathcal{R}e\{X(j\omega)\}$.

Note: You should perform all these calculations without explicitly evaluating $X(j\omega)$.

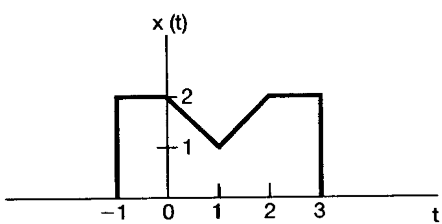

Figure P4.25

4.26. **(a)** Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t)$, $h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t)$, $h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t)$, $h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

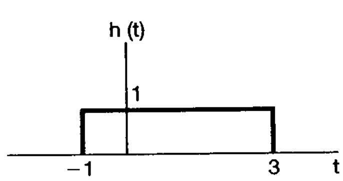

Figure P4.26

4.27. Consider the signals

$$x(t) = u(t-1) - 2u(t-2) + u(t-3)$$

and

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(t-kT),$$

where $T > 0$. Let a_k denote the Fourier series coefficients of $\tilde{x}(t)$, and let $X(j\omega)$ denote the Fourier transform of $x(t)$.

(a) Determine a closed-form expression for $X(j\omega)$.

(b) Determine an expression for the Fourier coefficients a_k and verify that $a_k = \frac{1}{T}X\left(j\frac{2\pi k}{T}\right)$.

4.28. **(a)** Let $x(t)$ have the Fourier transform $X(j\omega)$, and let $p(t)$ be periodic with fundamental frequency ω_0 and Fourier series representation

$$p(t) = \sum_{n=-\infty}^{+\infty} a_n e^{jn\omega_0 t}.$$

Determine an expression for the Fourier transform of

$$y(t) = x(t)p(t). \tag{P4.28–1}$$

(b) Suppose that $X(j\omega)$ is as depicted in Figure P4.28(a). Sketch the spectrum of $y(t)$ in eq. (P4.28–1) for each of the following choices of $p(t)$:

(i) $p(t) = \cos(t/2)$

(ii) $p(t) = \cos t$

(iii) $p(t) = \cos 2t$

(iv) $p(t) = (\sin t)(\sin 2t)$

(v) $p(t) = \cos 2t - \cos t$

(vi) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$

(vii) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n)$

(viii) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 4\pi n)$

(ix) $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - 2\pi n) - \frac{1}{2}\sum_{n=-\infty}^{+\infty} \delta(t - \pi n)$

(x) $p(t) =$ the periodic square wave shown in Figure P4.28(b).

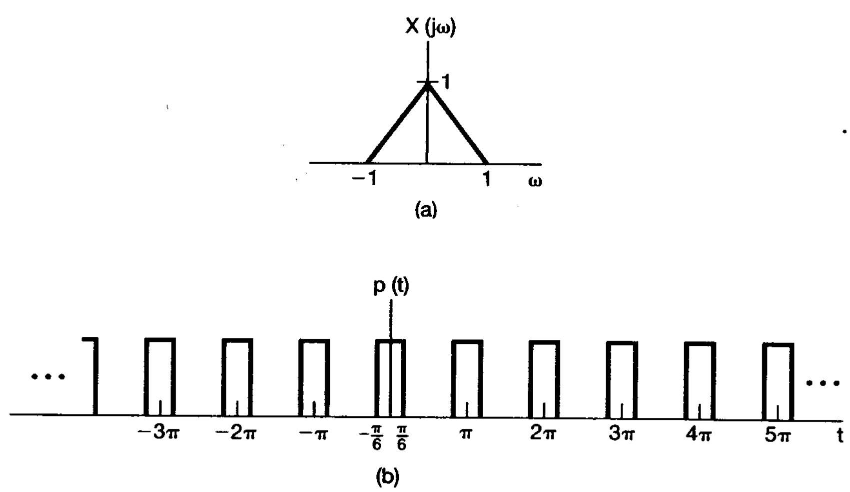

Figure P4.28

4.29. A real-valued continuous-time function $x(t)$ has a Fourier transform $X(j\omega)$ whose magnitude and phase are as illustrated in Figure P4.29(a).

The functions $x_a(t)$, $x_b(t)$, $x_c(t)$, and $x_d(t)$ have Fourier transforms whose magnitudes are identical to $X(j\omega)$, but whose phase functions differ, as shown in Figures P4.29(b)–(e). The phase functions $\sphericalangle X_a(j\omega)$ and $\sphericalangle X_b(j\omega)$ are formed by adding a linear phase to $\sphericalangle X(j\omega)$. The function $\sphericalangle X_c(j\omega)$ is formed by reflecting $\sphericalangle X(j\omega)$ about $\omega = 0$, and $\sphericalangle X_d(j\omega)$ is obtained by a combination of a reflection and an addition of a linear phase. Using the properties of Fourier transforms, determine the expressions for $x_a(t)$, $x_b(t)$, $x_c(t)$, and $x_d(t)$ in terms of $x(t)$.

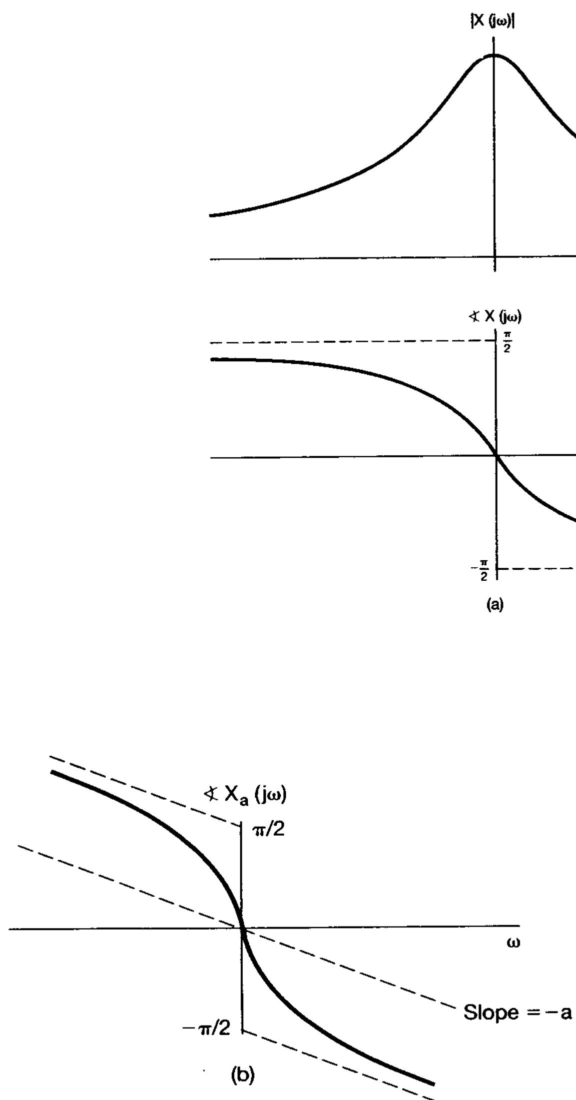

Figure P4.29

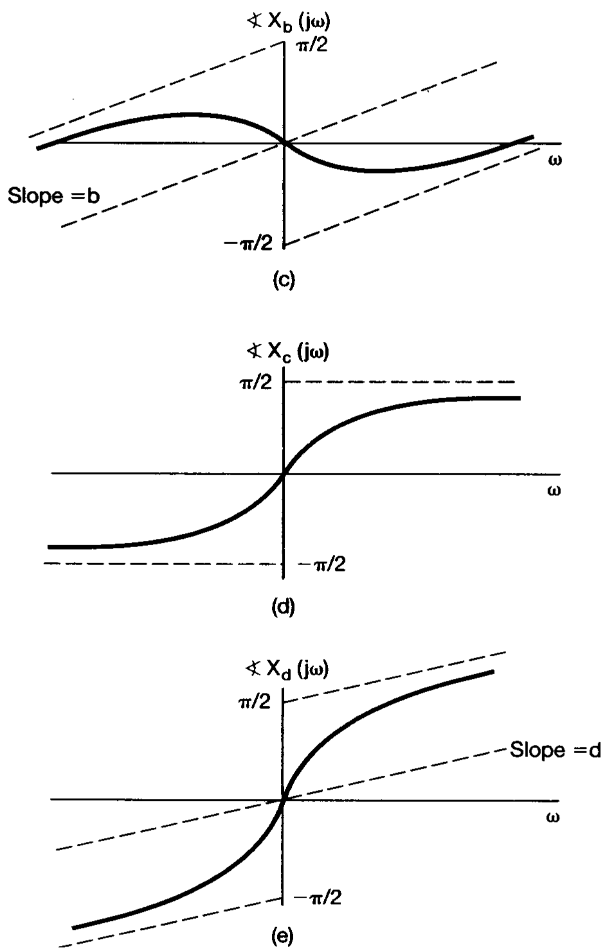

Figure P4.29 *Continued*

4.30. Suppose $g(t) = x(t)\cos t$ and the Fourier transform of the $g(t)$ is

$$G(j\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Determine $x(t)$.

(b) Specify the Fourier transform $X_1(j\omega)$ of a signal $x_1(t)$ such that

$$g(t) = x_1(t)\cos\left(\frac{2}{3}t\right).$$

4.31. **(a)** Show that the three LTI systems with impulse responses

$$h_1(t) = u(t),$$

$$h_2(t) = -2\delta(t) + 5e^{-2t}u(t),$$

and

$$h_3(t) = 2te^{-t}u(t)$$

all have the same response to $x(t) = \cos t$.

(b) Find the impulse response of another LTI system with the same response to $\cos t$.

This problem illustrates the fact that the response to $\cos t$ cannot be used to specify an LTI system uniquely.

4.32. Consider an LTI system S with impulse response

$$h(t) = \frac{\sin(4(t-1))}{\pi(t-1)}.$$

Determine the output of S for each of the following inputs:

(a) $x_1(t) = \cos(6t + \frac{\pi}{2})$

(b) $x_2(t) = \sum_{k=0}^{\infty}(\frac{1}{2})^k \sin(3kt)$

(c) $x_3(t) = \frac{\sin(4(t+1))}{\pi(t+1)}$

(d) $x_4(t) = (\frac{\sin 2t}{\pi t})^2$

4.33. The input and the output of a stable and causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) Find the impulse response of this system.

(b) What is the response of this system if $x(t) = te^{-2t}u(t)$?

(c) Repeat part (a) for the stable and causal LTI system described by the equation

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = 2\frac{d^2x(t)}{dt^2} - 2x(t)$$

4.34. A causal and stable LTI system S has the frequency response

$$H(j\omega) = \frac{j\omega + 4}{6 - \omega^2 + 5j\omega}.$$

(a) Determine a differential equation relating the input $x(t)$ and output $y(t)$ of S.
(b) Determine the impulse response $h(t)$ of S.
(c) What is the output of S when the input is

$$x(t) = e^{-4t}u(t) - te^{-4t}u(t)?$$

4.35. In this problem, we provide examples of the effects of nonlinear changes in phase.
(a) Consider the continuous-time LTI system with frequency response

$$H(j\omega) = \frac{a - j\omega}{a + j\omega},$$

where $a > 0$. What is the magnitude of $H(j\omega)$? What is $\measuredangle H(j\omega)$? What is the impulse response of this system?
(b) Determine the output of the system of part (a) with $a = 1$ when the input is

$$\cos(t/\sqrt{3}) + \cos t + \cos \sqrt{3}t.$$

Roughly sketch both the input and the output.

4.36. Consider an LTI system whose response to the input

$$x(t) = [e^{-t} + e^{-3t}]u(t)$$

is

$$y(t) = [2e^{-t} - 2e^{-4t}]u(t).$$

(a) Find the frequency response of this system.
(b) Determine the system's impulse response.
(c) Find the differential equation relating the input and the output of this system.

ADVANCED PROBLEMS

4.37. Consider the signal $x(t)$ in Figure P4.37.
(a) Find the Fourier transform $X(j\omega)$ of $x(t)$.
(b) Sketch the signal

$$\tilde{x}(t) = x(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(c) Find another signal $g(t)$ such that $g(t)$ is not the same as $x(t)$ and

$$\tilde{x}(t) = g(t) * \sum_{k=-\infty}^{\infty} \delta(t - 4k).$$

(d) Argue that, although $G(j\omega)$ is different from $X(j\omega)$, $G(j\frac{\pi k}{2}) = X(j\frac{\pi k}{2})$ for all integers k. You should not explicitly evaluate $G(j\omega)$ to answer this question.

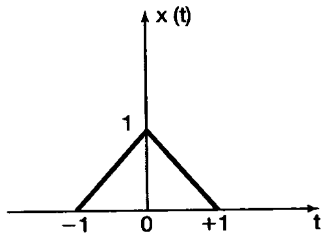

Figure P4.37

4.38. Let $x(t)$ be any signal with Fourier transform $X(j\omega)$. The frequency-shift property of the Fourier transform may be stated as

$$e^{j\omega_0 t}x(t) \stackrel{\mathfrak{F}}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

(a) Prove the frequency-shift property by applying the frequency shift to the analysis equation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

(b) Prove the frequency-shift property by utilizing the Fourier transform of $e^{j\omega_0 t}$ in conjunction with the multiplication property of the Fourier transform.

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

(b) Using the fact that

$$\mathfrak{F}\{\delta(t + B)\} = e^{jB\omega}$$

in conjunction with the result from part (a), show that

$$\mathfrak{F}\{e^{jBt}\} = 2\pi\,\delta(\omega - B).$$

4.40. Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t),\ a > 0,$$

is

$$\frac{1}{(a + j\omega)^n}.$$

4.41. In this problem, we derive the multiplication property of the continuous-time Fourier transform. Let $x(t)$ and $y(t)$ be two continuous-time signals with Fourier transforms $X(j\omega)$ and $Y(j\omega)$, respectively. Also, let $g(t)$ denote the inverse Fourier transform of $\frac{1}{2\pi}\{X(j\omega) * Y(j\omega)\}$.

(a) Show that

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega - \theta))e^{j\omega t}\,d\omega\right]d\theta.$$

(b) Show that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(j(\omega - \theta))e^{j\omega t}\,d\omega = e^{j\theta t}y(t).$$

(c) Combine the results of parts (a) and (b) to conclude that

$$g(t) = x(t)y(t).$$

4.42. Let

$$g_1(t) = \{[\cos(\omega_0 t)]x(t)\} * h(t) \quad \text{and} \quad g_2(t) = \{[\sin(\omega_0 t)]x(t)\} * h(t),$$

where

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk100t}$$

is a real-valued periodic signal and $h(t)$ is the impulse response of a stable LTI system.

(a) Specify a value for ω_0 and any necessary constraints on $H(j\omega)$ to ensure that

$$g_1(t) = \mathcal{R}e\{a_5\} \quad \text{and} \quad g_2(t) = \mathcal{I}m\{a_5\}.$$

(b) Give an example of $h(t)$ such that $H(j\omega)$ satisfies the constraints you specified in part (a).

4.43. Let

$$g(t) = x(t)\cos^2 t * \frac{\sin t}{\pi t}.$$

Assuming that $x(t)$ is real and $X(j\omega) = 0$ for $|\omega| \geq 1$, show that there exists an LTI system S such that

$$x(t) \xrightarrow{S} g(t).$$

4.44. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the equation

$$\frac{dy(t)}{dt} + 10y(t) = \int_{-\infty}^{+\infty} x(\tau)z(t-\tau)\,d\tau - x(t),$$

where $z(t) = e^{-t}u(t) + 3\,\delta(t)$.

(a) Find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of this system.

(b) Determine the impulse response of the system.

4.45. In the discussion in Section 4.3.7 of Parseval's relation for continuous-time signals, we saw that

$$\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\,d\omega.$$

This says that the total energy of the signal can be obtained by integrating $|X(j\omega)|^2$ over all frequencies. Now consider a real-valued signal $x(t)$ processed by the ideal bandpass filter $H(j\omega)$ shown in Figure P4.45. Express the energy in the output signal $y(t)$ as an integration over frequency of $|X(j\omega)|^2$. For Δ sufficiently small so that $|X(j\omega)|$ is approximately constant over a frequency interval of width Δ, show that the energy in the output $y(t)$ of the bandpass filter is approximately proportional to $\Delta|X(j\omega_0)|^2$.

On the basis of the foregoing result, $\Delta|X(j\omega_0)|^2$ is proportional to the energy in the signal in a bandwidth Δ around the frequency ω_0. For this reason, $|X(j\omega)|^2$ is often referred to as the *energy-density spectrum* of the signal $x(t)$.

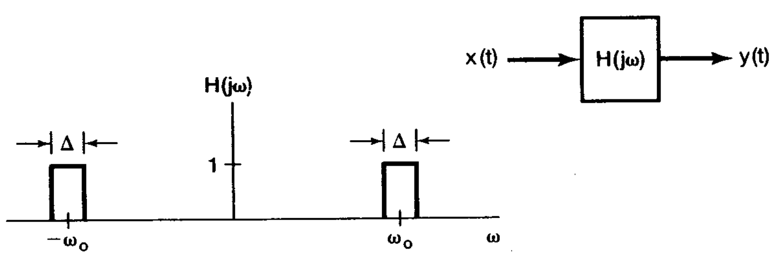

Figure P4.45

4.46. In Section 4.5.1, we discussed the use of amplitude modulation with a complex exponential carrier to implement a bandpass filter. The specific system was shown in Figure 4.26, and if only the real part of $f(t)$ is retained, the equivalent bandpass filter is that shown in Figure 4.30.

In Figure P4.46, we indicate an implementation of a bandpass filter using sinusoidal modulation and lowpass filters. Show that the output $y(t)$ of the system is identical to that which would be obtained by retaining only $\Re e\{f(t)\}$ in Figure 4.26.

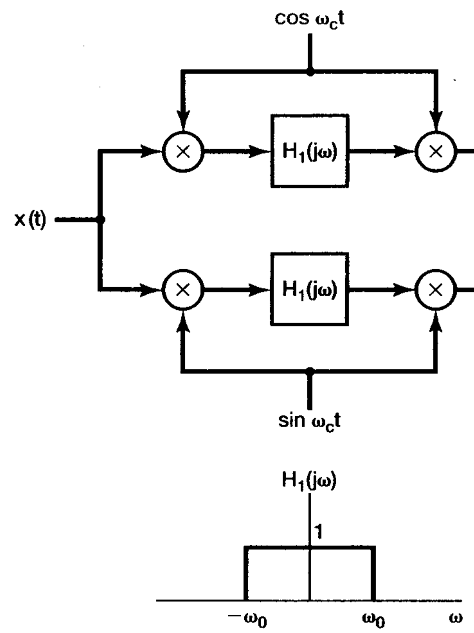

Figure P4.46

4.47. An important property of the frequency response $H(j\omega)$ of a continuous-time LTI system with a real, causal impulse response $h(t)$ is that $H(j\omega)$ is completely specified by its real part, $\mathcal{R}e\{H(j\omega)\}$. The current problem is concerned with deriving and examining some of the implications of this property, which is generally referred to as *real-part sufficiency*.

(a) Prove the property of real-part sufficiency by examining the signal $h_e(t)$, which is the even part of $h(t)$. What is the Fourier transform of $h_e(t)$? Indicate how $h(t)$ can be recovered from $h_e(t)$.

(b) If the real part of the frequency response of a causal system is

$$\mathcal{R}e\{H(j\omega)\} = \cos\omega,$$

what is $h(t)$?

(c) Show that $h(t)$ can be recovered from $h_o(t)$, the odd part of $h(t)$, for every value of t except $t = 0$. Note that if $h(t)$ does not contain any singularities $[\delta(t), u_1(t), u_2(t), \text{etc.}]$ at $t = 0$, then the frequency response

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,dt$$

will not change if $h(t)$ is set to some arbitrary finite value at the single point $t = 0$. Thus, in this case, show that $H(j\omega)$ is also completely specified by its imaginary part.

EXTENSION PROBLEMS

4.48. Let us consider a system with a real and causal impulse response $h(t)$ that does not have any singularities at $t = 0$. In Problem 4.47, we saw that either the real or the imaginary part of $H(j\omega)$ completely determines $H(j\omega)$. In this problem we derive an explicit relationship between $H_R(j\omega)$ and $H_I(j\omega)$, the real and imaginary parts of $H(j\omega)$.

(a) To begin, note that since $h(t)$ is causal,

$$h(t) = h(t)u(t), \tag{P4.48–1}$$

except perhaps at $t = 0$. Now, since $h(t)$ contains no singularities at $t = 0$, the Fourier transforms of both sides of eq. (P4.48–1) must be identical. Use this fact, together with the multiplication property, to show that

$$H(j\omega) = \frac{1}{j\pi}\int_{-\infty}^{+\infty} \frac{H(j\eta)}{\omega - \eta} d\eta. \tag{P4.48–2}$$

Use eq. (P4.48–2) to determine an expression for $H_R(j\omega)$ in terms of $H_I(j\omega)$ and one for $H_I(j\omega)$ in terms of $H_R(j\omega)$.

(b) The operation

$$y(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{x(\tau)}{t - \tau} d\tau \tag{P4.48–3}$$

is called the *Hilbert transform*. We have just seen that the real and imaginary parts of the transform of a real, causal impulse response $h(t)$ can be determined from one another using the Hilbert transform.

Now consider eq. (P4.48–3), and regard $y(t)$ as the output of an LTI system with input $x(t)$. Show that the frequency response of this system is

$$H(j\omega) = \begin{cases} -j, & \omega > 0 \\ j, & \omega < 0 \end{cases}.$$

(c) What is the Hilbert transform of the signal $x(t) = \cos 3t$?

4.49. Let $H(j\omega)$ be the frequency response of a continuous-time LTI system, and suppose that $H(j\omega)$ is real, even, and positive. Also, assume that

$$\max_{\omega}\{H(j\omega)\} = H(0).$$

(a) Show that:

(i) The impulse response, $h(t)$, is real.

(ii) $\max\{|h(t)|\} = h(0)$.

Hint: If $f(t, \omega)$ is a complex function of two variables, then

$$\left|\int_{-\infty}^{+\infty} f(t, \omega)\, d\omega\right| \leq \int_{-\infty}^{+\infty} |f(t, \omega)|\, d\omega.$$

(b) One important concept in system analysis is the *bandwidth* of an LTI system. There are many different mathematical ways in which to define bandwidth, but they are related to the qualitative and intuitive idea that a system with frequency response $G(j\omega)$ essentially "stops" signals of the form $e^{j\omega t}$ for values of ω where $G(j\omega)$ vanishes or is small and "passes" those complex exponentials in the band of frequency where $G(j\omega)$ is not small. The width of this band is the bandwidth. These ideas will be made much clearer in Chapter 6, but for now we will consider a special definition of bandwidth for those systems with frequency responses that have the properties specified previously for $H(j\omega)$. Specifically, one definition of the bandwidth B_w of such a system is the width of the rectangle of height $H(j0)$ that has an area equal to the area under $H(j\omega)$. This is illustrated in Figure P4.49(a). Note that since $H(j0) = \max_\omega H(j\omega)$, the frequencies within the band indicated in the figure are those for which $H(j\omega)$ is largest. The exact choice of the width in the figure is, of course, a bit arbitrary, but we have chosen one definition that allows us to compare different systems and to make precise a very important relationship between time and frequency.

What is the bandwidth of the system with frequency response

$$H(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}?$$

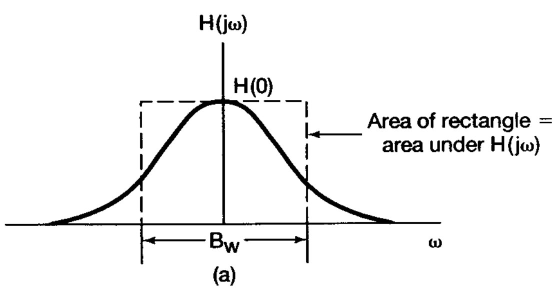

Figure P4.49a

(c) Find an expression for the bandwidth B_w in terms of $H(j\omega)$.

(d) Let $s(t)$ denote the step response of the system set out in part (a). An important measure of the speed of response of a system is the *rise time*, which, like the bandwidth, has a qualitative definition, leading to many possible mathematical definitions, one of which we will use. Intuitively, the rise time of a system is a measure of how fast the step response rises from zero to its final value,

$$s(\infty) = \lim_{t\to\infty} s(t).$$

Thus, the smaller the rise time, the faster is the response of the system. For the system under consideration in this problem, we will define the rise time as

$$t_r = \frac{s(\infty)}{h(0)}.$$

Since

$$s'(t) = h(t),$$

and also because of the property that $h(0) = \max_t h(t)$, t_r is the time it would take to go from zero to $s(\infty)$ while maintaining the maximum rate of change of $s(t)$. This is illustrated in Figure P4.49(b).

Find an expression for t_r in terms of $H(j\omega)$.

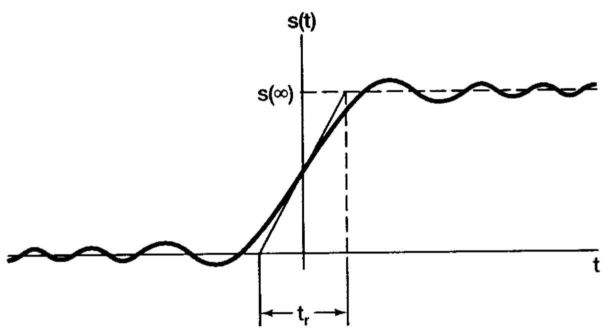

Figure P4.49b

(e) Combine the results of parts (c) and (d) to show that

$$B_w t_r = 2\pi. \qquad \text{(P4.49–1)}$$

Thus, we *cannot* independently specify both the rise time and the bandwidth of our system. For example, eq. (P4.49–1) implies that, if we want a fast system (t_r small), the system must have a large bandwidth. This is a fundamental trade-off that is of central importance in many problems of system design.

4.50. In Problems 1.45 and 2.67, we defined and examined several of the properties and uses of correlation functions. In the current problem, we examine the properties of such functions in the frequency domain. Let $x(t)$ and $y(t)$ be two real signals. Then the cross-correlation function of $x(t)$ and $y(t)$ is defined as

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)\,d\tau.$$

Similarly, we can define $\phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$. [The last two of these are called the autocorrelation functions of the signals $x(t)$ and $y(t)$, respectively.] Let $\Phi_{xy}(j\omega)$, $\Phi_{yx}(j\omega)$, $\Phi_{xx}(j\omega)$, and $\Phi_{yy}(j\omega)$ denote the Fourier transforms of $\phi_{xy}(t)$, $\phi_{yx}(t)$, $\phi_{xx}(t)$, and $\phi_{yy}(t)$, respectively.

(a) What is the relationship between $\Phi_{xy}(j\omega)$ and $\Phi_{yx}(j\omega)$?

(b) Find an expression for $\Phi_{xy}(j\omega)$ in terms of $X(j\omega)$ and $Y(j\omega)$.

(c) Show that $\Phi_{xx}(j\omega)$ is real and nonnegative for every ω.

(d) Suppose now that $x(t)$ is the input to an LTI system with a real-valued impulse response and with frequency response $H(j\omega)$ and that $y(t)$ is the output. Find expressions for $\Phi_{xy}(j\omega)$ and $\Phi_{yy}(j\omega)$ in terms of $\Phi_{xx}(j\omega)$ and $H(j\omega)$.

(e) Let $x(t)$ be as is illustrated in Figure P4.50, and let the LTI system impulse response be $h(t) = e^{-at}u(t)$, $a > 0$. Compute $\Phi_{xx}(j\omega)$, $\Phi_{xy}(j\omega)$, and $\Phi_{yy}(j\omega)$ using the results of parts (a)–(d).

(f) Suppose that we are given the following Fourier transform of a function $\phi(t)$:

$$\Phi(j\omega) = \frac{\omega^2 + 100}{\omega^2 + 25}.$$

Find the impulse responses of *two* causal, stable LTI systems that have autocorrelation functions equal to $\phi(t)$. Which one of these has a causal, stable inverse?

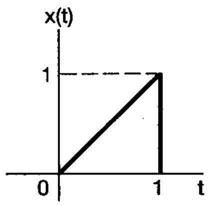

Figure P4.50

4.51. (a) Consider two LTI systems with impulse responses $h(t)$ and $g(t)$, respectively, and suppose that these systems are inverses of one another. Suppose also that the systems have frequency responses denoted by $H(j\omega)$ and $G(j\omega)$, respectively. What is the relationship between $H(j\omega)$ and $G(j\omega)$?

(b) Consider the continuous-time LTI system with frequency response

$$H(j\omega) = \begin{cases} 1, & 2 < |\omega| < 3 \\ 0, & \text{otherwise} \end{cases}.$$

(i) Is it possible to find an input $x(t)$ to this system such that the output is as depicted in Figure P4.50? If so, find $x(t)$. If not, explain why not.

(ii) Is this system invertible? Explain your answer.

(c) Consider an auditorium with an echo problem. As discussed in Problem 2.64, we can model the acoustics of the auditorium as an LTI system with an impulse response consisting of an impulse train, with the kth impulse in the train corresponding to the kth echo. Suppose that in this particular case the impulse response is

$$h(t) = \sum_{k=0}^{\infty} e^{-kT}\,\delta(t - kT),$$

where the factor e^{-kT} represents the attenuation of the kth echo.

In order to make a high-quality recording from the stage, the effect of the echoes must be removed by performing some processing of the sounds sensed by the recording equipment. In Problem 2.64, we used convolutional techniques to consider one example of the design of such a processor (for a different acoustic model). In the current problem, we will use frequency-domain techniques. Specifically, let $G(j\omega)$ denote the frequency response of the LTI system to be used to process the sensed acoustic signal. Choose $G(j\omega)$ so that the echoes are completely removed and the resulting signal is a faithful reproduction of the original stage sounds.

(d) Find the differential equation for the inverse of the system with impulse response

$$h(t) = 2\delta(t) + u_1(t).$$

(e) Consider the LTI system initially at rest and described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = \frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t).$$

The inverse of this system is also initially at rest and described by a differential equation. Find the differential equation describing the inverse, and find the impulse responses $h(t)$ and $g(t)$ of the original system and its inverse.

4.52. Inverse systems frequently find application in problems involving imperfect measuring devices. For example, consider a device for measuring the temperature of a liquid. It is often reasonable to model such a device as an LTI system that, because of the response characteristics of the measuring element (e.g., the mercury in a thermometer), does not respond instantaneously to temperature changes. In particular, assume that the response of this device to a unit step in temperature is

$$s(t) = (1 - e^{-t/2})u(t). \tag{P4.52–1}$$

(a) Design a compensatory system that, when provided with the output of the measuring device, produces an output equal to the instantaneous temperature of the liquid.

(b) One of the problems that often arises in using inverse systems as compensators for measuring devices is that gross inaccuracies in the indicated temperature may occur if the actual output of the measuring device produces errors due to small, erratic phenomena in the device. Since there always are such sources of error in real systems, one must take them into account. To illustrate this, consider a measuring device whose overall output can be modeled as the sum of the response of the measuring device characterized by eq. (P4.52–1) and an interfering "noise" signal $n(t)$. Such a model is depicted in Figure P4.52(a), where we have also included the inverse system of part (a), which now has as its input the *overall* output of the measuring device. Suppose that $n(t) = \sin \omega t$. What is the contribution of $n(t)$ to the output of the inverse system, and how does this output change as ω is increased?

(c) The issue raised in part (b) is an important one in many applications of LTI system analysis. Specifically, we are confronted with the fundamental tradeoff between the speed of response of the system and the ability of the system to attenuate high-frequency interference. In part (b) we saw that this tradeoff implied that, by attempting to speed up the response of a measuring device (by means of an inverse system), we produced a system that would also amplify

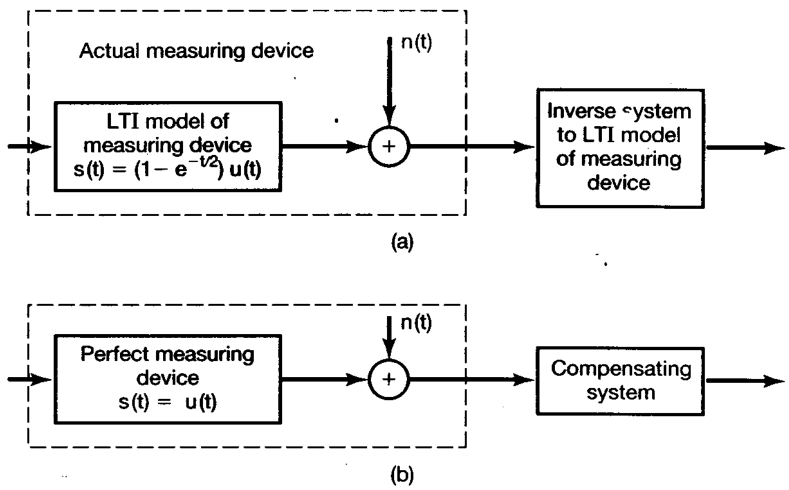

Figure P4.52

corrupting sinusoidal signals. To illustrate this concept further, consider a measuring device that responds instantaneously to changes in temperature, but that also is corrupted by noise. The response of such a system can be modeled, as depicted in Figure P4.52(b), as the sum of the response of a perfect measuring device and a corrupting signal $n(t)$. Suppose that we wish to design a compensatory system that will *slow down* the response to actual temperature variations, but also will attenuate the noise $n(t)$. Let the impulse response of this system be

$$h(t) = ae^{-at}u(t).$$

Choose a so that the overall system of Figure P4.52(b) responds as quickly as possible to a step change in temperature, subject to the constraint that the amplitude of the portion of the output due to the noise $n(t) = \sin 6t$ is no larger than 1/4.

4.53. As mentioned in the text, the techniques of Fourier analysis can be extended to signals having two independent variables. As their one-dimensional counterparts do in some applications, these techniques play an important role in other applications, such as image processing. In this problem, we introduce some of the elementary ideas of two-dimensional Fourier analysis.

Let $x(t_1, t_2)$ be a signal that depends upon two independent variables t_1 and t_2. The *two-dimensional Fourier transform* of $x(t_1, t_2)$ is defined as

$$X(j\omega_1, j\omega_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t_1, t_2)e^{-j(\omega_1 t_1 + \omega_2 t_2)}\,dt_1\,dt_2.$$

(a) Show that this double integral can be performed as two successive one-dimensional Fourier transforms, first in t_1 with t_2 regarded as fixed and then in t_2.

(b) Use the result of part (a) to determine the inverse transform—that is, an expression for $x(t_1, t_2)$ in terms of $X(j\omega_1, j\omega_2)$.

(c) Determine the two-dimensional Fourier transforms of the following signals:

(i) $x(t_1, t_2) = e^{-t_1+2t_2}u(t_1 - 1)u(2 - t_2)$

(ii) $x(t_1, t_2) = \begin{cases} e^{-|t_1|-|t_2|}, & \text{if } -1 < t_1 \leq 1 \text{ and } -1 \leq t_2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$

(iii) $x(t_1, t_2) = \begin{cases} e^{-|t_1|-|t_2|}, & \text{if } 0 \leq t_1 \leq 1 \text{ or } 0 \leq t_2 \leq 1 \text{ (or both)} \\ 0, & \text{otherwise} \end{cases}$

(iv) $x(t_1, t_2)$ as depicted in Figure P4.53.

(v) $e^{-|t_1+t_2|-|t_1-t_2|}$

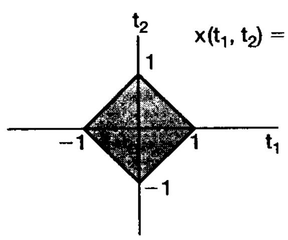

Figure P4.53

(d) Determine the signal $x(t_1, t_2)$ whose two-dimensional Fourier transform is

$$X(j\omega_1, j\omega_2) = \frac{2\pi}{4 + j\omega_1}\delta(\omega_2 - 2\omega_1).$$

(e) Let $x(t_1, t_2)$ and $h(t_1, t_2)$ be two signals with two-dimensional Fourier transforms $X(j\omega_1, j\omega_2)$ and $H(j\omega_1, j\omega_2)$, respectively. Determine the transforms of the following signals in terms of $X(j\omega_1, j\omega_2)$ and $H(j\omega_1, j\omega_2)$:

(i) $x(t_1 - T_1, t_2 - T_2)$

(ii) $x(at_1, bt_2)$

(iii) $y(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau_1, \tau_2)h(t_1 - \tau_1, t_2 - \tau_2)\,d\tau_1\,d\tau_2$