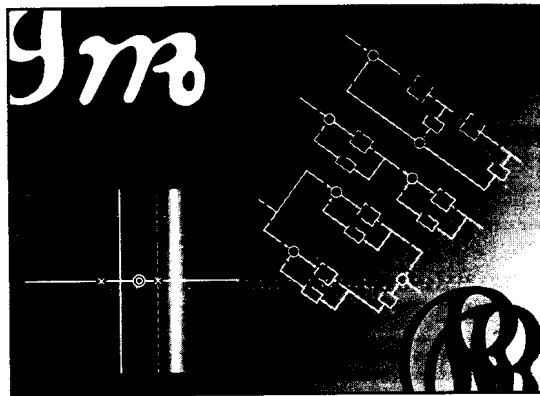


9

THE LAPLACE TRANSFORM



9.0 INTRODUCTION

In the preceding chapters, we have seen that the tools of Fourier analysis are extremely useful in the study of many problems of practical importance involving signals and LTI systems. This is due in large part to the fact that broad classes of signals can be represented as linear combinations of periodic complex exponentials and that complex exponentials are eigenfunctions of LTI systems. The continuous-time Fourier transform provides us with a representation for signals as linear combinations of complex exponentials of the form e^{st} with $s = j\omega$. However the eigenfunction property introduced in Section 3.2 and many of its consequences apply as well for arbitrary values of s and not only those values that are purely imaginary. This observation leads to a generalization of the continuous-time Fourier transform, known as the Laplace transform, which we develop in this chapter. In the next chapter we develop the corresponding discrete-time generalization known as the z -transform.

As we will see, the Laplace and z -transforms have many of the properties that make Fourier analysis so useful. Moreover, not only do these transforms provide additional tools and insights for signals and systems that can be analyzed using the Fourier transform, but they also can be applied in some very important contexts in which Fourier transforms cannot. For example Laplace and z -transforms can be applied to the analysis of many unstable systems and consequently play an important role in the investigation of the stability or instability of systems. This fact, combined with the algebraic properties that Laplace and z -transforms share with Fourier transforms, leads to a very important set of tools for system analysis and in particular for the analysis of feedback systems, which we develop in Chapter 11.

9.1 THE LAPLACE TRANSFORM

In Chapter 3, we saw that the response of a linear time-invariant system with impulse response $h(t)$ to a complex exponential input of the form e^{st} is

$$y(t) = H(s)e^{st}, \quad (9.1)$$

where

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt. \quad (9.2)$$

For s imaginary (i.e., $s = j\omega$), the integral in eq. (9.2) corresponds to the Fourier transform of $h(t)$. For general values of the complex variable s , it is referred to as the *Laplace transform* of the impulse response $h(t)$.

The Laplace transform of a general signal $x(t)$ is defined as¹

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st} dt, \quad (9.3)$$

and we note in particular that it is a function of the independent variable s corresponding to the complex variable in the exponent of e^{-st} . The complex variable s can be written as $s = \sigma + j\omega$, with σ and ω the real and imaginary parts, respectively. For convenience, we will sometimes denote the Laplace transform in operator form as $\mathcal{L}\{x(t)\}$ and denote the transform relationship between $x(t)$ and $X(s)$ as

$$x(t) \xleftrightarrow{\mathcal{L}} X(s). \quad (9.4)$$

When $s = j\omega$, eq. (9.3) becomes

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt, \quad (9.5)$$

which corresponds to the *Fourier transform* of $x(t)$; that is,

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}. \quad (9.6)$$

The Laplace transform also bears a straightforward relationship to the Fourier transform when the complex variable s is not purely imaginary. To see this relationship, consider $X(s)$ as specified in eq. (9.3) with s expressed as $s = \sigma + j\omega$, so that

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t} dt, \quad (9.7)$$

¹The transform defined by eq. (9.3) is often called the *bilateral Laplace transform*, to distinguish it from the unilateral Laplace transform, which we discuss in Section 9.9. The bilateral transform in eq. (9.3) involves an integration from $-\infty$ to $+\infty$, while the unilateral transform has a form similar to that in eq. (9.3), but with limits of integration from 0 to $+\infty$. As we are primarily concerned with the bilateral transform, we will omit the word "bilateral," except where it is needed in Section 9.9 to avoid ambiguity.

or

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt. \quad (9.8)$$

We recognize the right-hand side of eq. (9.8) as the Fourier transform of $x(t)e^{-\sigma t}$; that is, the Laplace transform of $x(t)$ can be interpreted as the Fourier transform of $x(t)$ after multiplication by a real exponential signal. The real exponential $e^{-\sigma t}$ may be decaying or growing in time, depending on whether σ is positive or negative.

To illustrate the Laplace transform and its relationship to the Fourier transform, let us consider the following example:

Example 9.1

Let the signal $x(t) = e^{-at}u(t)$. From Example 4.1, the Fourier transform $X(j\omega)$ converges for $a > 0$ and is given by

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-at}e^{-j\omega t} dt = \frac{1}{j\omega + a}, \quad a > 0. \quad (9.9)$$

From eq. (9.3), the Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt, \quad (9.10)$$

or, with $s = \sigma + j\omega$,

$$X(\sigma + j\omega) = \int_0^{\infty} e^{-(\sigma+a)t} e^{-j\omega t} dt. \quad (9.11)$$

By comparison with eq. (9.9) we recognize eq. (9.11) as the Fourier transform of $e^{-(\sigma+a)t}u(t)$, and thus,

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0, \quad (9.12)$$

or equivalently, since $s = \sigma + j\omega$ and $\sigma = \Re\{s\}$,

$$X(s) = \frac{1}{s + a}, \quad \Re\{s\} > -a. \quad (9.13)$$

That is,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s + a}, \quad \Re\{s\} > -a. \quad (9.14)$$

For example, for $a = 0$, $x(t)$ is the unit step with Laplace transform $X(s) = 1/s$, $\Re\{s\} > 0$.

We note, in particular, that just as the Fourier transform does not converge for all signals, the Laplace transform may converge for some values of $\Re\{s\}$ and not for others. In eq. (9.13), the Laplace transform converges only for $\sigma = \Re\{s\} > -a$. If a is positive,

then $X(s)$ can be evaluated at $\sigma = 0$ to obtain

$$X(0 + j\omega) = \frac{1}{j\omega + a}. \tag{9.15}$$

As indicated in eq. (9.6), for $\sigma = 0$ the Laplace transform is equal to the Fourier transform, as is evident in the preceding example by comparing eqs. (9.9) and (9.15). If a is negative or zero, the Laplace transform still exists, but the Fourier transform does not.

Example 9.2

For comparison with Example 9.1, let us consider as a second example the signal

$$x(t) = -e^{-at}u(-t). \tag{9.16}$$

Then

$$\begin{aligned} X(s) &= -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t) dt \\ &= -\int_{-\infty}^0 e^{-(s+a)t} dt, \end{aligned} \tag{9.17}$$

or

$$X(s) = \frac{1}{s+a}. \tag{9.18}$$

For convergence in this example, we require that $\Re\{s+a\} < 0$, or $\Re\{s\} < -a$; that is,

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad \Re\{s\} < -a. \tag{9.19}$$

Comparing eqs. (9.14) and (9.19), we see that the algebraic expression for the Laplace transform is identical for both of the signals considered in Examples 9.1 and 9.2. However, from the same equations, we also see that the set of values of s for which the expression is valid is very different in the two examples. This serves to illustrate the fact that, in specifying the Laplace transform of a signal, both the algebraic expression and the range of values of s for which this expression is valid are required. In general, the range of values of s for which the integral in eq.(9.3) converges is referred to as the *region of convergence* (which we abbreviate as ROC) of the Laplace transform. That is, the ROC consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges. We will have more to say about the ROC as we develop some insight into the properties of the Laplace transform.

A convenient way to display the ROC is shown in Figure 9.1. The variable s is a complex number, and in the figure we display the complex plane, generally referred to as the s -plane, associated with this complex variable. The coordinate axes are $\Re\{s\}$ along the horizontal axis and $\Im\{s\}$ along the vertical axis. The horizontal and vertical axes are sometimes referred to as the σ -axis and the $j\omega$ -axis, respectively. The shaded region in Figure 9.1(a) represents the set of points in the s -plane corresponding to the region of convergence for Example 9.1. The shaded region in Figure 9.1(b) indicates the region of convergence for Example 9.2.

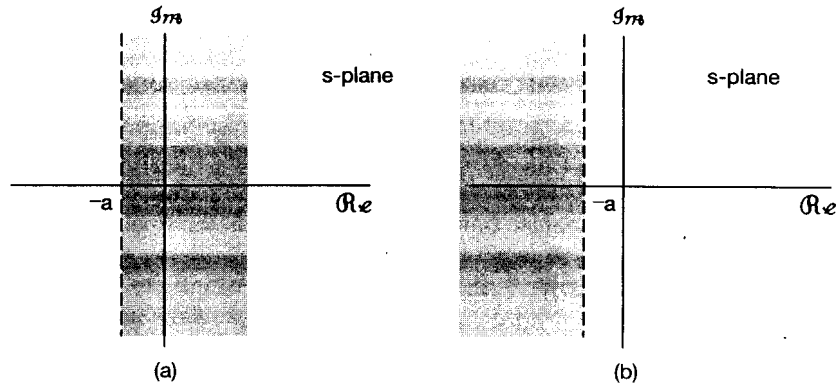


Figure 9.1 (a) ROC for Example 9.1; (b) ROC for Example 9.2.

Example 9.3

In this example, we consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t). \quad (9.20)$$

The algebraic expression for the Laplace transform is then

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)] e^{-st} dt \\ &= 3 \int_{-\infty}^{\infty} e^{-2t} e^{-st} u(t) dt - 2 \int_{-\infty}^{\infty} e^{-t} e^{-st} u(t) dt. \end{aligned} \quad (9.21)$$

Each of the integrals in eq. (9.21) is of the same form as the integral in eq. (9.10), and consequently, we can use the result in Example 9.1 to obtain

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}. \quad (9.22)$$

To determine the ROC we note that $x(t)$ is a sum of two real exponentials, and from eq. (9.21) we see that $X(s)$ is the sum of the Laplace transforms of each of the individual terms. The first term is the Laplace transform of $3e^{-2t}u(t)$ and the second term the Laplace transform of $-2e^{-t}u(t)$. From Example 9.1, we know that

$$\begin{aligned} e^{-t}u(t) &\xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, & \Re\{s\} > -1, \\ e^{-2t}u(t) &\xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, & \Re\{s\} > -2. \end{aligned}$$

The set of values of $\Re\{s\}$ for which the Laplace transforms of both terms converge is $\Re\{s\} > -1$, and thus, combining the two terms on the right-hand side of eq. (9.22), we obtain

$$3e^{-2t}u(t) - 2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{s-1}{s^2+3s+2}, \quad \Re\{s\} > -1. \quad (9.23)$$

Example 9.4

In this example, we consider a signal that is the sum of a real and a complex exponential:

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t). \quad (9.24)$$

Using Euler's relation, we can write

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t} \right] u(t),$$

and the Laplace transform of $x(t)$ then can be expressed as

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st} dt \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} e^{-(1-3j)t}u(t)e^{-st} dt \\ &+ \frac{1}{2} \int_{-\infty}^{\infty} e^{-(1+3j)t}u(t)e^{-st} dt. \end{aligned} \quad (9.25)$$

Each of the integrals in eq. (9.25) represents a Laplace transform of the type encountered in Example 9.1. It follows that

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \Re\{s\} > -2, \quad (9.26)$$

$$e^{-(1-3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1-3j)}, \quad \Re\{s\} > -1, \quad (9.27)$$

$$e^{-(1+3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1+3j)}, \quad \Re\{s\} > -1. \quad (9.28)$$

For all three Laplace transforms to converge simultaneously, we must have $\Re\{s\} > -1$. Consequently, the Laplace transform of $x(t)$ is

$$\frac{1}{s+2} + \frac{1}{2} \left(\frac{1}{s+(1-3j)} \right) + \frac{1}{2} \left(\frac{1}{s+(1+3j)} \right), \quad \Re\{s\} > -1, \quad (9.29)$$

or, with terms combined over a common denominator,

$$e^{-2t}u(t) + e^{-t}(\cos 3t)u(t) \xleftrightarrow{\mathcal{L}} \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s + 2)}, \quad \Re\{s\} > -1. \quad (9.30)$$

In each of the four preceding examples, the Laplace transform is rational, i.e., it is a ratio of polynomials in the complex variable s , so that

$$X(s) = \frac{N(s)}{D(s)}, \quad (9.31)$$

where $N(s)$ and $D(s)$ are the numerator polynomial and denominator polynomial, respectively. As suggested by Examples 9.3 and 9.4, $X(s)$ will be rational whenever $x(t)$ is a linear combination of real or complex exponentials. As we will see in Section 9.7, rational

transforms also arise when we consider LTI systems specified in terms of linear constant-coefficient differential equations. Except for a scale factor, the numerator and denominator polynomials in a rational Laplace transform can be specified by their roots; thus, marking the locations of the roots of $N(s)$ and $D(s)$ in the s -plane and indicating the ROC provides a convenient pictorial way of describing the Laplace transform. For example, in Figure 9.2(a) we show the s -plane representation of the Laplace transform of Example 9.3, with the location of each root of the denominator polynomial in eq. (9.23) indicated with "×" and the location of the root of the numerator polynomial in eq. (9.23) indicated with "o." The corresponding plot of the roots of the numerator and denominator polynomials for the Laplace transform in Example 9.4 is given in Figure 9.2(b). The region of convergence for each of these examples is shaded in the corresponding plot.

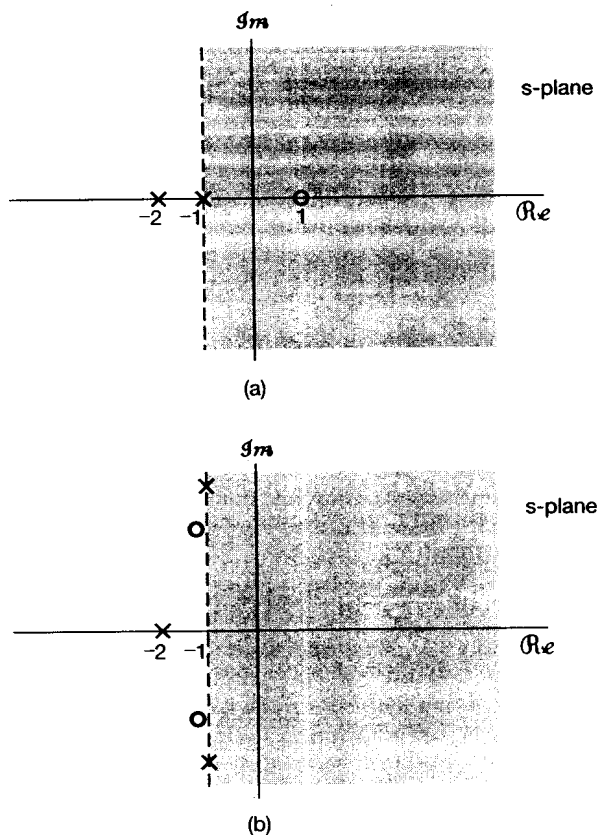


Figure 9.2 s -plane representation of the Laplace transforms for (a) Example 9.3 and (b) Example 9.4. Each × in these figures marks the location of a pole of the corresponding Laplace transform—i.e., a root of the denominator. Similarly, each o marks a zero—i.e., a root of the numerator. The shaded regions indicate the ROCs.

For rational Laplace transforms, the roots of the numerator polynomial are commonly referred to as the *zeros* of $X(s)$, since, for those values of s , $X(s) = 0$. The roots of the denominator polynomial are referred to as the *poles* of $X(s)$, and for those values of s , $X(s)$ is infinite. The poles and zeros of $X(s)$ in the finite s -plane completely characterize the algebraic expression for $X(s)$ to within a scale factor. The representation of $X(s)$ through its poles and zeros in the s -plane is referred to as the *pole-zero plot* of $X(s)$.

However, as we saw in Examples 9.1 and 9.2, knowledge of the algebraic form of $X(s)$ does not by itself identify the ROC for the Laplace transform. That is, a complete specification, to within a scale factor, of a rational Laplace transform consists of the pole-zero plot of the transform, together with its ROC (which is commonly shown as a shaded region in the s -plane, as in Figures 9.1 and 9.2).

Also, while they are not needed to specify the algebraic form of a rational transform $X(s)$, it is sometimes convenient to refer to poles or zeros of $X(s)$ at infinity. Specifically, if the order of the denominator polynomial is greater than the order of the numerator polynomial, then $X(s)$ will become zero as s approaches infinity. Conversely, if the order of the numerator polynomial is greater than the order of the denominator, then $X(s)$ will become unbounded as s approaches infinity. This behavior can be interpreted as zeros or poles at infinity. For example, the Laplace transform in eq. (9.23) has a denominator of order 2 and a numerator of order only 1, so in this case $X(s)$ has one zero at infinity. The same is true for the transform in eq. (9.30), in which the numerator is of order 2 and the denominator is of order 3. In general, if the order of the denominator exceeds the order of the numerator by k , $X(s)$ will have k zeros at infinity. Similarly, if the order of the numerator exceeds the order of the denominator by k , $X(s)$ will have k poles at infinity.

Example 9.5

Let

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t). \quad (9.32)$$

The Laplace transform of the second and third terms on the right-hand side of eq. (9.32) can be evaluated from Example 9.1. The Laplace transform of the unit impulse can be evaluated directly as

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)e^{-st} dt = 1, \quad (9.33)$$

which is valid for any value of s . That is, the ROC of $\mathcal{L}\{\delta(t)\}$ is the entire s -plane. Using this result, together with the Laplace transforms of the other two terms in eq. (9.32), we obtain

$$X(s) = 1 - \frac{4}{3} \frac{1}{s+1} + \frac{1}{3} \frac{1}{s-2}, \quad \text{Re}\{s\} > 2, \quad (9.34)$$

or

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}, \quad \text{Re}\{s\} > 2, \quad (9.35)$$

where the ROC is the set of values of s for which the Laplace transforms of all three terms in $x(t)$ converge. The pole-zero plot for this example is shown in Figure 9.3, together with the ROC. Also, since the degrees of the numerator and denominator of $X(s)$ are equal, $X(s)$ has neither poles nor zeros at infinity.

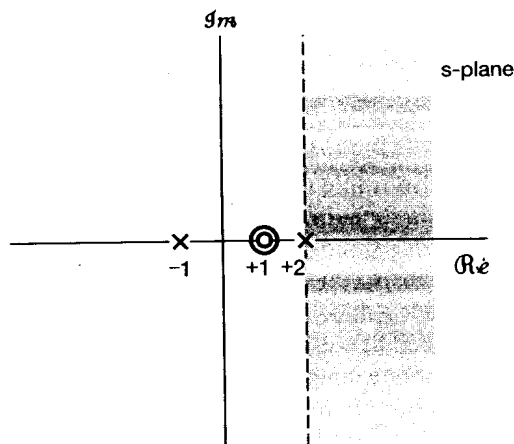


Figure 9.3 Pole-zero plot and ROC for Example 9.5.

Recall from eq. (9.6) that, for $s = j\omega$, the Laplace transform corresponds to the Fourier transform. However, if the ROC of the Laplace transform does not include the $j\omega$ -axis, (i.e., if $\Re\{s\} = 0$), then the Fourier transform does not converge. As we see from Figure 9.3, this, in fact, is the case for Example 9.5, which is consistent with the fact that the term $(1/3)e^{2t}u(t)$ in $x(t)$ does not have a Fourier transform. Note also in this example that the two zeros in eq. (9.35) occur at the same value of s . In general, we will refer to the *order* of a pole or zero as the number of times it is repeated at a given location. In Example 9.5 there is a second-order zero at $s = 1$ and two first-order poles, one at $s = -1$, the other at $s = 2$. In this example the ROC lies to the right of the rightmost pole. In general, for rational Laplace transforms, there is a close relationship between the locations of the poles and the possible ROCs that can be associated with a given pole-zero plot. Specific constraints on the ROC are closely associated with time-domain properties of $x(t)$. In the next section, we explore some of these constraints and properties.

9.2 THE REGION OF CONVERGENCE FOR LAPLACE TRANSFORMS

In the preceding section, we saw that a complete specification of the Laplace transform requires not only the algebraic expression for $X(s)$, but also the associated region of convergence. As evidenced by Examples 9.1 and 9.2, two very different signals can have identical algebraic expressions for $X(s)$, so that their Laplace transforms are distinguishable *only* by the region of convergence. In this section, we explore some specific constraints on the ROC for various classes of signals. As we will see, an understanding of these constraints often permits us to specify implicitly or to reconstruct the ROC from knowledge of only the algebraic expression for $X(s)$ and certain general characteristics of $x(t)$ in the time domain.

Property 1: The ROC of $X(s)$ consists of strips parallel to the $j\omega$ -axis in the s -plane.

The validity of this property stems from the fact that the ROC of $X(s)$ consists of those values of $s = \sigma + j\omega$ for which the Fourier transform of $x(t)e^{-\sigma t}$ converges. That

is, the ROC of the Laplace transform of $x(t)$ consists of those values of s for which $x(t)e^{-\sigma t}$ is absolutely integrable:²

$$\int_{-\infty}^{+\infty} |x(t)|e^{-\sigma t} dt < \infty. \tag{9.36}$$

Property 1 then follows, since this condition depends only on σ , the real part of s .

Property 2: For rational Laplace transforms, the ROC does not contain any poles.

Property 2 is easily observed in all the examples studied thus far. Since $X(s)$ is infinite at a pole, the integral in eq. (9.3) clearly does not converge at a pole, and thus the ROC cannot contain values of s that are poles.

Property 3: If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s -plane.

The intuition behind this result is suggested in Figures 9.4 and 9.5. Specifically, a finite-duration signal has the property that it is zero outside an interval of finite duration, as illustrated in Figure 9.4. In Figure 9.5(a), we have shown $x(t)$ of Figure 9.4 multiplied by a decaying exponential, and in Figure 9.5(b) the same signal multiplied by a growing exponential

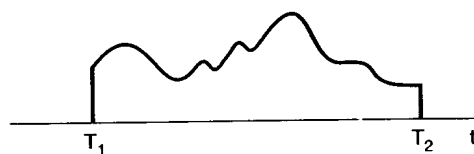


Figure 9.4 Finite-duration signal.

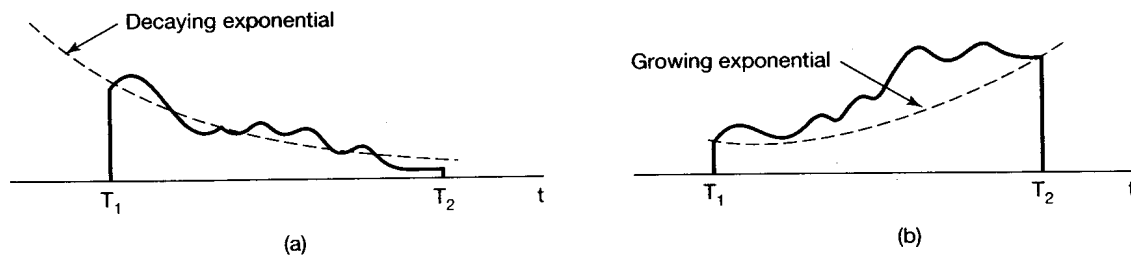


Figure 9.5 (a) Finite-duration signal of Figure 9.4 multiplied by a decaying exponential; (b) finite-duration signal of Figure 9.4 multiplied by a growing exponential.

²For a more thorough and formal treatment of Laplace transforms and their mathematical properties, including convergence, see E. D. Rainville, *The Laplace Transform: An Introduction* (New York: Macmillan, 1963), and R. V. Churchill and J. W. Brown, *Complex Variables and Applications* (5th ed.) (New York: McGraw-Hill, 1990). Note that the condition of absolute integrability is one of the Dirichlet conditions introduced in Section 4.1 in the context of our discussion of the convergence of Fourier transforms.

exponential. Since the interval over which $x(t)$ is nonzero is finite, the exponential weighting is never unbounded, and consequently, it is reasonable that the integrability of $x(t)$ not be destroyed by this exponential weighting.

A more formal verification of Property 3 is as follows: Suppose that $x(t)$ is absolutely integrable, so that

$$\int_{T_1}^{T_2} |x(t)| dt < \infty. \quad (9.37)$$

For $s = \sigma + j\omega$ to be in the ROC, we require that $x(t)e^{-\sigma t}$ be absolutely integrable, i.e.,

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t} dt < \infty. \quad (9.38)$$

Eq. (9.37) verifies that s is in the ROC when $\Re\{s\} = \sigma = 0$. For $\sigma > 0$, the maximum value of $e^{-\sigma t}$ over the interval on which $x(t)$ is nonzero is $e^{-\sigma T_1}$, and thus we can write

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t} dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| dt. \quad (9.39)$$

Since the right-hand side of eq.(9.39) is bounded, so is the left-hand side; therefore, the s -plane for $\Re\{s\} > 0$ must also be in the ROC. By a similar argument, if $\sigma < 0$, then

$$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t} dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)| dt, \quad (9.40)$$

and again, $x(t)e^{-\sigma t}$ is absolutely integrable. Thus, the ROC includes the entire s -plane.

Example 9.6

Let

$$x(t) = \begin{cases} e^{-at}, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}. \quad (9.41)$$

Then

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a} [1 - e^{-(s+a)T}]. \quad (9.42)$$

Since in this example $x(t)$ is of finite length, it follows from Property 3 that the ROC is the entire s -plane. In the form of eq. (9.42), $X(s)$ would appear to have a pole at $s = -a$, which, from Property 2, would be inconsistent with an ROC that consists of the entire s -plane. In fact, however, in the algebraic expression in eq. (9.42), both numerator and denominator are zero at $s = -a$, and thus, to determine $X(s)$ at $s = -a$, we can use L'hôpital's rule to obtain

$$\lim_{s \rightarrow -a} X(s) = \lim_{s \rightarrow -a} \left[\frac{\frac{d}{ds}(1 - e^{-(s+a)T})}{\frac{d}{ds}(s+a)} \right] = \lim_{s \rightarrow -a} T e^{-aT} e^{-sT},$$

so that

$$X(-a) = T. \quad (9.43)$$

It is important to recognize that, to ensure that the exponential weighting is bounded over the interval in which $x(t)$ is nonzero, the preceding discussion relies heavily on the fact that $x(t)$ is of finite duration. In the next two properties, we consider modifications of the result in Property 3 when $x(t)$ is of finite extent in only the positive-time or negative-time direction.

Property 4: If $x(t)$ is right sided, and if the line $\Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\Re\{s\} > \sigma_0$ will also be in the ROC.

A *right-sided* signal is a signal for which $x(t) = 0$ prior to some finite time T_1 , as illustrated in Figure 9.6. It is possible that, for such a signal, there is no value of s for which the Laplace transform will converge. One example is the signal $x(t) = e^{t^2} u(t)$. However, suppose that the Laplace transform converges for some value of σ , which we denote by σ_0 . Then

$$\int_{-\infty}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty, \quad (9.44)$$

or equivalently, since $x(t)$ is right sided,

$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty. \quad (9.45)$$

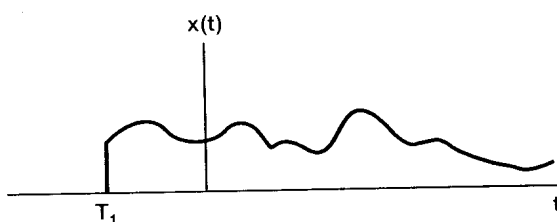


Figure 9.6 Right-sided signal.

Then if $\sigma_1 > \sigma_0$, it must also be true that $x(t)e^{-\sigma_1 t}$ is absolutely integrable, since $e^{-\sigma_1 t}$ decays faster than $e^{-\sigma_0 t}$ as $t \rightarrow +\infty$, as illustrated in Figure 9.7. Formally, we can say that with $\sigma_1 > \sigma_0$,

$$\begin{aligned} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t} dt &= \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt \\ &\leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} dt. \end{aligned} \quad (9.46)$$

Since T_1 is finite, it follows from eq. (9.45) that the right side of the inequality in eq. (9.46) is finite, and hence, $x(t)e^{-\sigma_1 t}$ is absolutely integrable.

Note that in the preceding argument we explicitly rely on the fact that $x(t)$ is right sided, so that, although with $\sigma_1 > \sigma_0$, $e^{-\sigma_1 t}$ diverges faster than $e^{-\sigma_0 t}$ as $t \rightarrow -\infty$, $x(t)e^{-\sigma_1 t}$ cannot grow without bound in the negative-time direction, since $x(t) = 0$ for

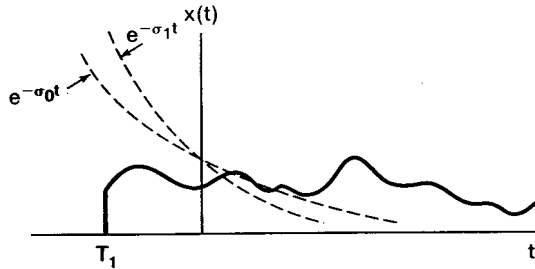


Figure 9.7 If $x(t)$ is right sided and $x(t)e^{-\sigma_0 t}$ is absolutely integrable, then $x(t)e^{-\sigma_1 t}$, $\sigma_1 > \sigma_0$, will also be absolutely integrable.

$t < T_1$. Also, in this case, if a point s is in the ROC, then all the points to the right of s , i.e., all points with larger real parts, are in the ROC. For this reason, the ROC in this case is commonly referred to as a *right-half plane*.

Property 5: If $x(t)$ is left sided, and if the line $\Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\Re\{s\} < \sigma_0$ will also be in the ROC.

A *left-sided* signal is a signal for which $x(t) = 0$ after some finite time T_2 , as illustrated in Figure 9.8. The argument and intuition behind this property are exactly analogous to the argument and intuition behind Property 4. Also, for a left-sided signal, the ROC is commonly referred to as a *left-half plane*, as if a point s is in the ROC, then all points to the left of s are in the ROC.

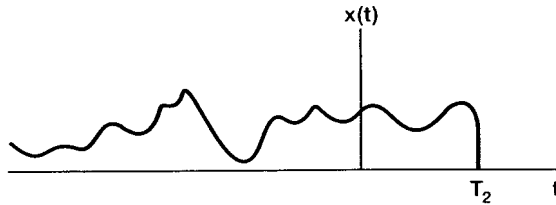


Figure 9.8 Left-sided signal.

Property 6: If $x(t)$ is two sided, and if the line $\Re\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s -plane that includes the line $\Re\{s\} = \sigma_0$.

A *two-sided* signal is a signal that is of infinite extent for both $t > 0$ and $t < 0$, as illustrated in Figure 9.9(a). For such a signal, the ROC can be examined by choosing an arbitrary time T_0 and dividing $x(t)$ into the sum of a right-sided signal $x_R(t)$ and a left-sided signal $x_L(t)$, as indicated in Figures 9.9(b) and 9.9(c). The Laplace transform of $x(t)$ converges for values of s for which the transforms of *both* $x_R(t)$ and $x_L(t)$ converge. From Property 4, the ROC of $\mathcal{L}\{x_R(t)\}$ consists of a half-plane $\Re\{s\} > \sigma_R$ for some value σ_R , and from Property 5, the ROC of $\mathcal{L}\{x_L(t)\}$ consists of a half-plane $\Re\{s\} < \sigma_L$ for some value σ_L . The ROC of $\mathcal{L}\{x(t)\}$ is then the overlap of these two half-planes, as indicated in Figure 9.10. This assumes, of course, that $\sigma_R < \sigma_L$, so that there is some overlap. If this is not the case, then even if the Laplace transforms of $x_R(t)$ and $x_L(t)$ individually exist, the Laplace transform of $x(t)$ does not.

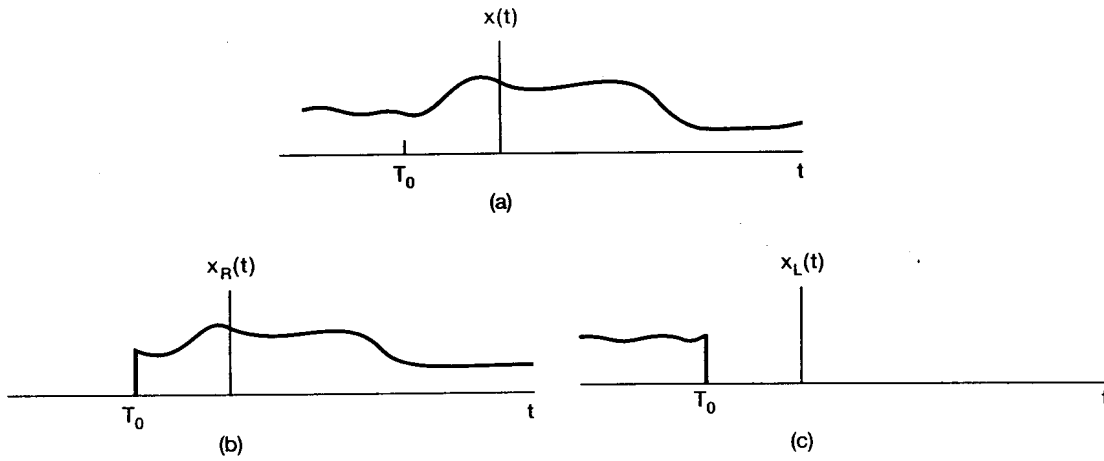


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided signal: (a) two-sided signal $x(t)$; (b) the right-sided signal equal to $x(t)$ for $t > T_0$ and equal to 0 for $t < T_0$; (c) the left-sided signal equal to $x(t)$ for $t < T_0$ and equal to 0 for $t > T_0$.

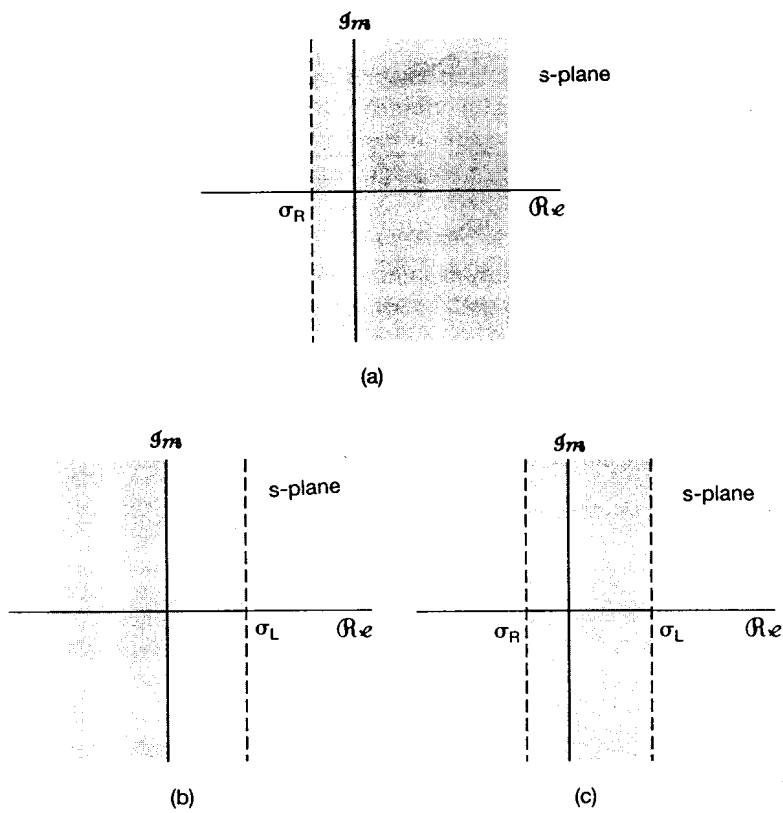


Figure 9.10 (a) ROC for $x_R(t)$ in Figure 9.9; (b) ROC for $x_L(t)$ in Figure 9.9; (c) the ROC for $x(t) = x_R(t) + x_L(t)$, assuming that the ROCs in (a) and

Example 9.7

Let

$$x(t) = e^{-b|t|}, \quad (9.47)$$

as illustrated in Figure 9.11 for both $b > 0$ and $b < 0$. Since this is a two-sided signal, let us divide it into the sum of a right-sided and left-sided signal; that is,

$$x(t) = e^{-bt}u(t) + e^{+bt}u(-t). \quad (9.48)$$

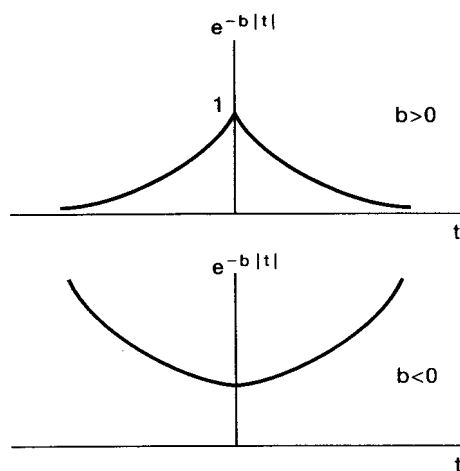


Figure 9.11 Signal $x(t) = e^{-b|t|}$ for both $b > 0$ and $b < 0$.

From Example 9.1,

$$e^{-bt}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+b}, \quad \Re\{s\} > -b, \quad (9.49)$$

and from Example 9.2,

$$e^{+bt}u(-t) \xleftrightarrow{\mathcal{L}} \frac{-1}{s-b}, \quad \Re\{s\} < +b. \quad (9.50)$$

Although the Laplace transforms of each of the individual terms in eq. (9.48) have a region of convergence, there is no *common* region of convergence if $b \leq 0$, and thus, for those values of b , $x(t)$ has no Laplace transform. If $b > 0$, the Laplace transform of $x(t)$ is

$$e^{-b|t|} \xleftrightarrow{\mathcal{L}} \frac{1}{s+b} - \frac{1}{s-b} = \frac{-2b}{s^2 - b^2}, \quad -b < \Re\{s\} < +b. \quad (9.51)$$

The corresponding pole-zero plot is shown in Figure 9.12, with the shading indicating the ROC.

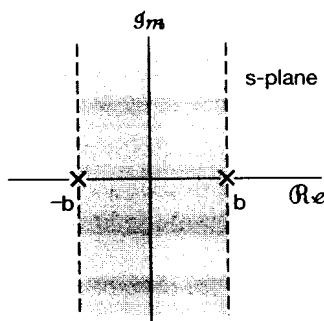


Figure 9.12 Pole-zero plot and ROC for Example 9.7.

A signal either does not have a Laplace transform or falls into one of the four categories covered by Properties 3 through 6. Thus, for any signal with a Laplace transform, the ROC *must* be the entire s -plane (for finite-length signals), a left-half plane (for left-sided signals), a right-half plane (for right-sided signals), or a single strip (for two-sided signals). In all the examples that we have considered, the ROC has the additional property that in each direction (i.e., $\Re\{s\}$ increasing and $\Re\{s\}$ decreasing) it is bounded by poles or extends to infinity. In fact, this is *always* true for rational Laplace transforms:

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

A formal argument establishing this property is somewhat involved, but its validity is essentially a consequence of the facts that a signal with a rational Laplace transform consists of a linear combination of exponentials and, from Examples 9.1 and 9.2, that the ROC for the transform of individual terms in this linear combination must have the property. As a consequence of Property 7, together with Properties 4 and 5, we have

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the region in the s -plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s -plane to the left of the leftmost pole.

To illustrate how different ROCs can be associated with the same pole-zero pattern, let us consider the following example:

Example 9.8

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad (9.52)$$

with the associated pole-zero pattern in Figure 9.13(a). As indicated in Figures 9.13(b)–(d), there are three possible ROCs that can be associated with this algebraic expression, corresponding to three distinct signals. The signal associated with the pole-zero pattern in Figure 9.13(b) is right sided. Since the ROC includes the $j\omega$ -axis, the Fourier

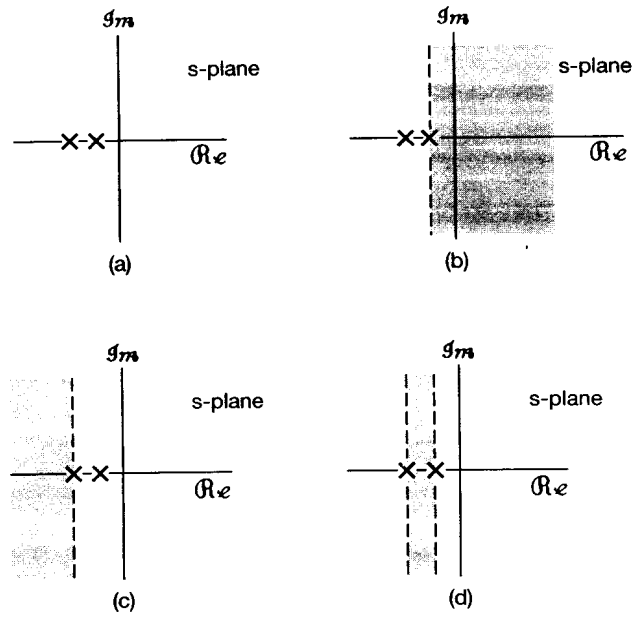


Figure 9.13 (a) Pole-zero pattern for Example 9.8; (b) ROC corresponding to a right-sided sequence; (c) ROC corresponding to a left-sided sequence; (d) ROC corresponding to a two-sided sequence.

transform of this signal converges. Figure 9.13(c) corresponds to a left-sided signal and Figure 9.13(d) to a two-sided signal. Neither of these two signals have Fourier transforms, since their ROCs do not include the $j\omega$ -axis.

9.3 THE INVERSE LAPLACE TRANSFORM

In Section 9.1 we discussed the interpretation of the Laplace transform of a signal as the Fourier transform of an exponentially weighted version of the signal; that is, with s expressed as $s = \sigma + j\omega$, the Laplace transform of a signal $x(t)$ is

$$X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} x(t)e^{-\sigma t} e^{-j\omega t} dt \quad (9.53)$$

for values of $s = \sigma + j\omega$ in the ROC. We can invert this relationship using the inverse Fourier transform as given in eq. (4.9). We have

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{j\omega t} d\omega, \quad (9.54)$$

or, multiplying both sides by $e^{\sigma t}$, we obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} d\omega. \quad (9.55)$$

That is, we can recover $x(t)$ from its Laplace transform evaluated for a set of values of $s = \sigma + j\omega$ in the ROC, with σ fixed and ω varying from $-\infty$ to $+\infty$. We can highlight this and gain additional insight into recovering $x(t)$ from $X(s)$ by changing the variable of integration in eq. (9.55) from ω to s and using the fact that σ is constant, so that $ds = j d\omega$. The result is the basic inverse Laplace transform equation:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds. \quad (9.56)$$

This equation states that $x(t)$ can be represented as a weighted integral of complex exponentials. The contour of integration in eq. (9.56) is the straight line in the s -plane corresponding to all points s satisfying $\Re\{s\} = \sigma$. This line is parallel to the $j\omega$ -axis. Furthermore, we can choose any such line in the ROC—i.e., we can choose any value of σ such that $X(\sigma + j\omega)$ converges. The formal evaluation of the integral for a general $X(s)$ requires the use of contour integration in the complex plane, a topic that we will not consider here. However, for the class of rational transforms, the inverse Laplace transform can be determined without directly evaluating eq. (9.56) by using the technique of partial-fraction expansion in a manner similar to that used in Chapter 4 to determine the inverse Fourier transform. Basically, the procedure consists of expanding the rational algebraic expression into a linear combination of lower order terms.

For example, assuming no multiple-order poles, and assuming that the order of the denominator polynomial is greater than the order of the numerator polynomial, we can expand $X(s)$ in the form

$$X(s) = \sum_{i=1}^m \frac{A_i}{s + a_i}. \quad (9.57)$$

From the ROC of $X(s)$, the ROC of each of the individual terms in eq. (9.57) can be inferred, and then, from Examples 9.1 and 9.2, the inverse Laplace transform of each of these terms can be determined. There are two possible choices for the inverse transform of each term $A_i/(s + a_i)$ in the equation. If the ROC is to the right of the pole at $s = -a_i$, then the inverse transform of this term is $A_i e^{-a_i t} u(t)$, a right-sided signal. If the ROC is to the left of the pole at $s = -a_i$, then the inverse transform of the term is $-A_i e^{-a_i t} u(-t)$, a left-sided signal. Adding the inverse transforms of the individual terms in eq. (9.57) then yields the inverse transform of $X(s)$. The details of this procedure are best presented through a number of examples.

Example 9.9

Let

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \Re\{s\} > -1. \quad (9.58)$$

To obtain the inverse Laplace transform, we first perform a partial-fraction expansion to obtain

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}. \quad (9.59)$$

As discussed in the appendix, we can evaluate the coefficients A and B by multiplying both sides of eq. (9.59) by $(s + 1)(s + 2)$ and then equating coefficients of equal powers of s on both sides. Alternatively, we can use the relation

$$A = [(s + 1)X(s)]|_{s=-1} = 1, \quad (9.60)$$

$$B = [(s + 2)X(s)]|_{s=-2} = -1. \quad (9.61)$$

Thus, the partial-fraction expansion for $X(s)$ is

$$X(s) = \frac{1}{s + 1} - \frac{1}{s + 2}. \quad (9.62)$$

From Examples 9.1 and 9.2, we know that there are two possible inverse transforms for a transform of the form $1/(s + a)$, depending on whether the ROC is to the left or the right of the pole. Consequently, we need to determine which ROC to associate with each of the individual first-order terms in eq. (9.62). This is done by reference to the properties of the ROC developed in Section 9.2. Since the ROC for $X(s)$ is $\Re\{s\} > -1$, the ROC for the individual terms in the partial-fraction expansion of eq. (9.62) includes $\Re\{s\} > -1$. The ROC for each term can then be extended to the left or right (or both) to be bounded by a pole or infinity. This is illustrated in Figure 9.14. Figure 9.14(a) shows the pole-zero plot and ROC for $X(s)$, as specified in eq. (9.58). Figure 9.14(b) and 9.14(c) represent the individual terms in the partial-fraction expansion in eq. (9.62). The ROC for the sum is indicated with lighter shading. For the term represented by Figure 9.14(c), the ROC for the sum can be extended to the left as shown, so that it is bounded by a pole.

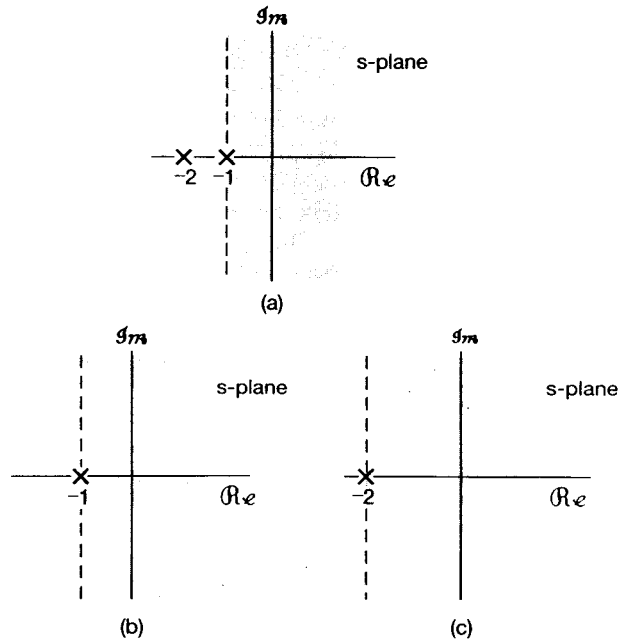


Figure 9.14 Construction of the ROCs for the individual terms in the partial-fraction expansion of $X(s)$ in Example 9.8: (a) pole-zero plot and ROC for $X(s)$; (b) pole at $s = -1$ and its ROC; (c) pole at $s = -2$ and its ROC.

Since the ROC is to the right of both poles, the same is true for each of the individual terms, as can be seen in Figures 9.14(b) and (c). Consequently, from Property 8 in the preceding section, we know that each of these terms corresponds to a right-sided signal. The inverse transform of the individual terms in eq. (9.62) can then be obtained by reference to Example 9.1:

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \quad \Re\{s\} > -1, \quad (9.63)$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \Re\{s\} > -2. \quad (9.64)$$

We thus obtain

$$[e^{-t} - e^{-2t}]u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \quad \Re\{s\} > -1. \quad (9.65)$$

Example 9.10

Let us now suppose that the algebraic expression for $X(s)$ is again given by eq. (9.58), but that the ROC is now the left-half plane $\Re\{s\} < -2$. The partial-fraction expansion for $X(s)$ relates only to the algebraic expression, so eq. (9.62) is still valid. With this new ROC, however, the ROC is to the *left* of both poles and thus, the same must be true for each of the two terms in the equation. That is, the ROC for the term corresponding to the pole at $s = -1$ is $\Re\{s\} < -1$, while the ROC for the term with pole at $s = -2$ is $\Re\{s\} < -2$. Then, from Example 9.2,

$$-e^{-t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1}, \quad \Re\{s\} < -1, \quad (9.66)$$

$$-e^{-2t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \Re\{s\} < -2, \quad (9.67)$$

so that

$$x(t) = [-e^{-t} + e^{-2t}]u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \quad \Re\{s\} < -2. \quad (9.68)$$

Example 9.11

Finally, suppose that the ROC of $X(s)$ in eq. (9.58) is $-2 < \Re\{s\} < -1$. In this case, the ROC is to the left of the pole at $s = -1$ so that this term corresponds to the left-sided signal in eq. (9.66), while the ROC is to the right of the pole at $s = -2$ so that this term corresponds to the right-sided signal in eq. (9.64). Combining these, we find that

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+1)(s+2)}, \quad -2 < \Re\{s\} < -1. \quad (9.69)$$

As discussed in the appendix, when $X(s)$ has multiple-order poles, or when the denominator is not of higher degree than the numerator, the partial-fraction expansion of $X(s)$ will include other terms in addition to the first-order terms considered in Examples 9.9–9.11. In Section 9.5, after discussing properties of the Laplace transform, we develop some other Laplace transform pairs that, in conjunction with the properties, allow us to extend the inverse transform method outlined in Example 9.9 to arbitrary rational transforms.

9.4 GEOMETRIC EVALUATION OF THE FOURIER TRANSFORM FROM THE POLE-ZERO PLOT

As we saw in Section 9.1, the Fourier transform of a signal is the Laplace transform evaluated on the $j\omega$ -axis. In this section we discuss a procedure for geometrically evaluating the Fourier transform and, more generally, the Laplace transform at any set of values from the pole-zero pattern associated with a rational Laplace transform. To develop the procedure, let us first consider a Laplace transform with a single zero [i.e., $X(s) = s - a$], which we evaluate at a specific value of s , say, $s = s_1$. The algebraic expression $s_1 - a$ is the sum of two complex numbers, s_1 and $-a$, each of which can be represented as a vector in the complex plane, as illustrated in Figure 9.15. The vector representing the complex number $s_1 - a$ is then the vector sum of s_1 and $-a$, which we see in the figure to be a vector from the zero at $s = a$ to the point s_1 . The value of $X(s_1)$ then has a magnitude that is the length of this vector and an angle that is the angle of the vector relative to the real axis. If $X(s)$ instead has a single pole at $s = a$ [i.e., $X(s) = 1/(s - a)$], then the denominator would be represented by the same vector sum of s_1 and $-a$, and the value of $X(s_1)$ would have a magnitude that is the *reciprocal* of the length of the vector from the pole to $s = s_1$ and an angle that is the *negative* of the angle of the vector with the real axis.

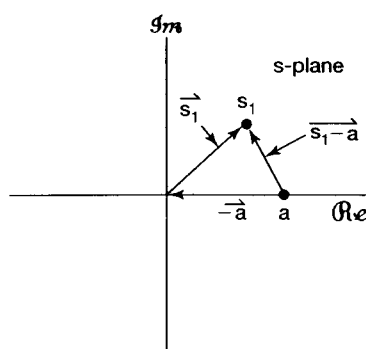


Figure 9.15 Complex plane representation of the vectors s_1 , a , and $s_1 - a$ representing the complex numbers s_1 , a , and $s_1 - a$, respectively.

A more general rational Laplace transform consists of a product of pole and zero terms of the form discussed in the preceding paragraph; that is, it can be factored into the form

$$X(s) = M \frac{\prod_{i=1}^R (s - \beta_i)}{\prod_{j=1}^P (s - \alpha_j)} \quad (9.70)$$

To evaluate $X(s)$ at $s = s_1$, each term in the product is represented by a vector from the zero or pole to the point s_1 . The magnitude of $X(s_1)$ is then the magnitude of the scale factor M , times the product of the lengths of the zero vectors (i.e., the vectors from the zeros to s_1) divided by the product of the lengths of the pole vectors (i.e., the vectors from the poles to s_1). The angle of the complex number $X(s_1)$ is the sum of the angles of the zero vectors minus the sum of the angles of the pole vectors. If the scale factor M in eq. (9.70) is negative, an additional angle of π would be included. If $X(s)$ has a multiple pole or zero

(or both), corresponding to some of the α_j 's being equal to each other or some of the β_i 's being equal to each other (or both), the lengths and angles of the vectors from each of these poles or zeros must be included a number of times equal to the order of the pole or zero.

Example 9.12

Let

$$X(s) = \frac{1}{s + \frac{1}{2}}, \quad \text{Re}\{s\} > -\frac{1}{2}. \tag{9.71}$$

The Fourier transform is $X(s)|_{s=j\omega}$. For this example, then, the Fourier transform is

$$X(j\omega) = \frac{1}{j\omega + 1/2}. \tag{9.72}$$

The pole-zero plot for $X(s)$ is shown in Figure 9.16. To determine the Fourier transform graphically, we construct the pole vector as indicated. The magnitude of the Fourier transform at frequency ω is the reciprocal of the length of the vector from the pole to the point $j\omega$ on the imaginary axis. The phase of the Fourier transform is the negative of the angle of the vector. Geometrically, from Figure 9.16, we can write

$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2} \tag{9.73}$$

and

$$\angle X(j\omega) = -\tan^{-1} 2\omega. \tag{9.74}$$

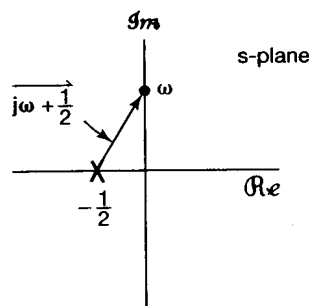


Figure 9.16 Pole-zero plot for Example 9.12. $|X(j\omega)|$ is the reciprocal of the length of the vector shown, and $\angle X(j\omega)$ is the negative of the angle of the vector.

Often, part of the value of the geometric determination of the Fourier transform lies in its usefulness in obtaining an approximate view of the overall characteristics of the transform. For example, in Figure 9.16, it is readily evident that the length of the pole vector monotonically increases with increasing ω , and thus, the magnitude of the Fourier

transform will monotonically *decrease* with increasing ω . The ability to draw general conclusions about the behavior of the Fourier transform from the pole-zero plot is further illustrated by a consideration of general first- and second-order systems.

9.4.1 First-Order Systems

As a generalization of Example 9.12, let us consider the class of first-order systems that was discussed in some detail in Section 6.5.1. The impulse response for such a system is

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t), \quad (9.75)$$

and its Laplace transform is

$$H(s) = \frac{1}{s\tau + 1}, \quad \Re\{s\} > -\frac{1}{\tau}. \quad (9.76)$$

The pole-zero plot is shown in Figure 9.17. Note from the figure that the length of the pole vector is minimal for $\omega = 0$ and increases monotonically as ω increases. Also, the angle of the pole increases monotonically from 0 to $\pi/2$ as ω increases from 0 to ∞ .

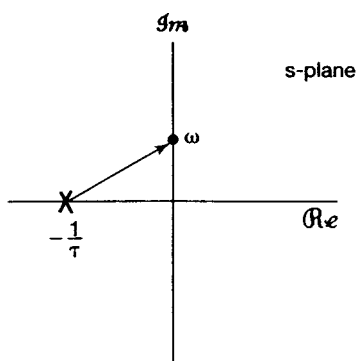


Figure 9.17 Pole-zero plot for first-order system of eq. (9.76).

From the behavior of the pole vector as ω varies, it is clear that the magnitude of the frequency response $H(j\omega)$ monotonically decreases as ω increases, while $\angle H(j\omega)$ monotonically decreases from 0 to $-\pi/2$, as shown in the Bode plots for this system in Figure 9.18. Note also that when $\omega = 1/\tau$, the real and imaginary parts of the pole vector are equal, yielding a value of the magnitude of the frequency response that is reduced by a factor of $\sqrt{2}$, or approximately 3 dB, from its maximum at $\omega = 0$ and a value of $\pi/4$ for the angle of the frequency response. This is consistent with our examination of first-order systems in Section 6.5.1, where we noted that $\omega = 1/\tau$ is often referred to as the 3-dB point or the break frequency—i.e., the frequency at which the straight-line approximation of the Bode plot of $|H(j\omega)|$ has a break in its slope. As we also saw in Section 6.5.1, the time constant τ controls the speed of response of first-order systems, and we now see that

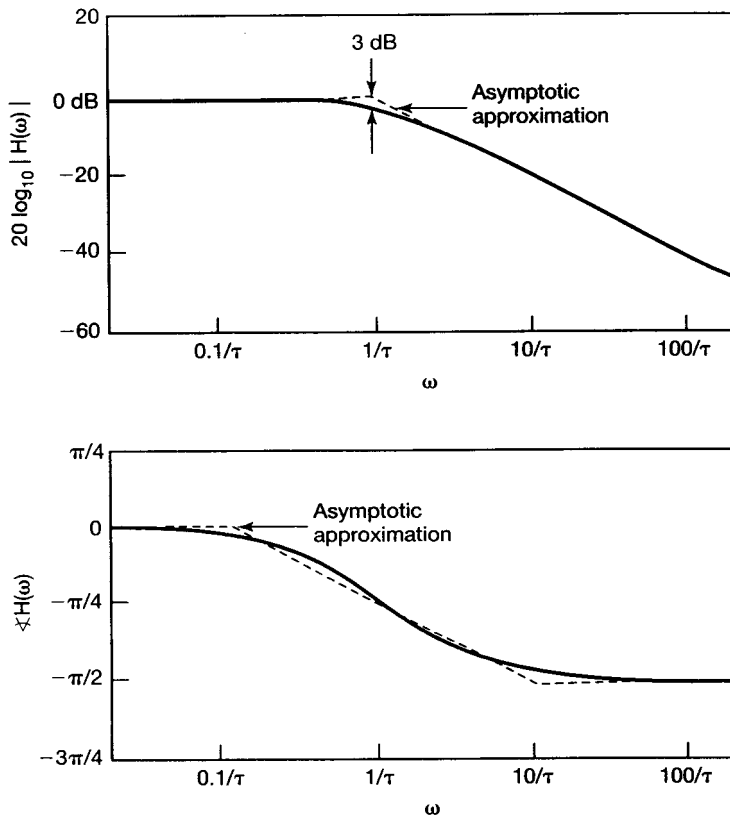


Figure 9.18 Frequency response for a first-order system.

the pole of such a system at $s = -1/\tau$ is on the negative real axis, at a distance to the origin that is the reciprocal of the time constant.

From our graphical interpretation, we can also see how changing the time constant or, equivalently, the position of the pole of $H(s)$ changes the characteristics of a first-order system. In particular, as the pole moves farther into the left-half plane, the break frequency and, hence, the effective cutoff frequency of the system increases. Also, from eq. (9.75) and from Figure 6.19, we see that this same movement of the pole to the left corresponds to a decrease in the time constant τ , resulting in a faster decay of the impulse response and a correspondingly faster rise time in the step response. This relationship between the real part of the pole locations and the speed of the system response holds more generally; that is, poles farther away from the $j\omega$ -axis are associated with faster response terms in the impulse response.

9.4.2 Second-Order Systems

Let us next consider the class of second-order systems, which was discussed in some detail in Section 6.5.2. The impulse response and frequency response for the system, originally

given in eqs. (6.37) and (6.33), respectively, are

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t), \quad (9.77)$$

where

$$\begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}, \\ M &= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}, \end{aligned}$$

and

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}. \quad (9.78)$$

The Laplace transform of the impulse response is

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}. \quad (9.79)$$

For $\zeta > 1$, c_1 and c_2 are real and thus both poles lie on the real axis, as indicated in Figure 9.19(a). The case of $\zeta > 1$ is essentially a product of two first-order terms, as in Section 9.4.1. Consequently, in this case $|H(j\omega)|$ decreases monotonically as $|\omega|$ increases, while $\angle H(j\omega)$ varies from 0 at $\omega = 0$ to $-\pi$ as $\omega \rightarrow \infty$. This can be verified from Figure 9.19(a) by observing that the length of the vector from each of the two poles to the point $s = j\omega$ increases monotonically as ω increases from 0, and the angle of each of these vectors increases from 0 to $\pi/2$ as ω increases from 0 to ∞ . Note also that as ζ increases, one pole moves closer to the $j\omega$ -axis, indicative of a term in the impulse response that decays more slowly, and the other pole moves farther into the left-half plane, indicative of a term in the impulse response that decays more rapidly. Thus, for large values of ζ , it is the pole close to the $j\omega$ -axis that dominates the system response for large time. Similarly, from a consideration of the pole vectors for $\zeta \gg 1$, as indicated in Figure 9.19(b), for low frequencies the length and angle of the vector for the pole close to the $j\omega$ -axis are much more sensitive to changes in ω than the length and angle of the vector for the pole far from the $j\omega$ -axis. Hence, we see that for low frequencies, the characteristics of the frequency response are influenced principally by the pole close to the $j\omega$ -axis.

For $0 < \zeta < 1$, c_1 and c_2 are complex, so that the pole-zero plot is that shown in Figure 9.19(c). Correspondingly, the impulse response and step response have oscillatory parts. We note that the two poles occur in complex conjugate locations. In fact, as we discuss in Section 9.5.5, the complex poles (and zeros) for a real-valued signal always occur in complex conjugate pairs. From the figure—particularly when ζ is small, so that the poles are close to the $j\omega$ -axis—as ω approaches $\omega_n\sqrt{1 - \zeta^2}$, the behavior of the frequency response is dominated by the pole vector in the second quadrant, and in

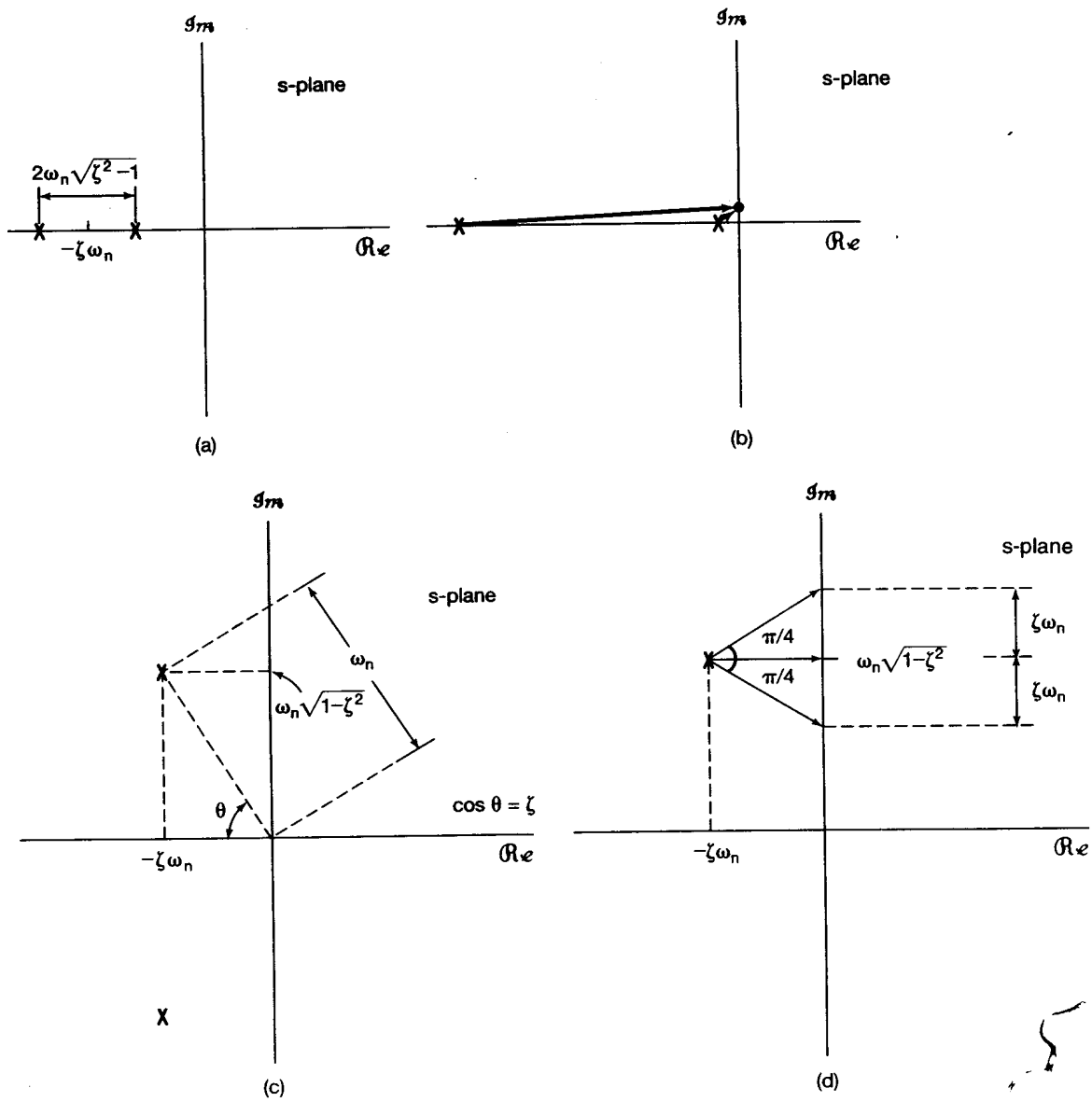


Figure 9.19 (a) Pole-zero plot for a second-order system with $\zeta > 1$; (b) pole vectors for $\zeta \gg 1$; (c) pole-zero plot for a second-order system with $0 < \zeta < 1$; (d) pole vectors for $0 < \zeta < 1$ and for $\omega = \omega_n\sqrt{1-\zeta^2}$ and $\omega = \omega_n\sqrt{1-\zeta^2} \pm \zeta\omega_n$.

particular, the length of that pole vector has a minimum at $\omega = \omega_n \sqrt{1 - \zeta^2}$. Thus, qualitatively, we would expect the magnitude of the frequency response to exhibit a peak in the vicinity of that frequency. Because of the presence of the other pole, the peak will occur not exactly at $\omega = \omega_n \sqrt{1 - \zeta^2}$, but at a frequency slightly less than this. A careful sketch of the magnitude of the frequency response is shown in Figure 9.20(a) for $\omega_n = 1$ and several values of ζ where the expected behavior in the vicinity of the poles is clearly evident. This is consistent with our analysis of second-order systems in Section 6.5.2.

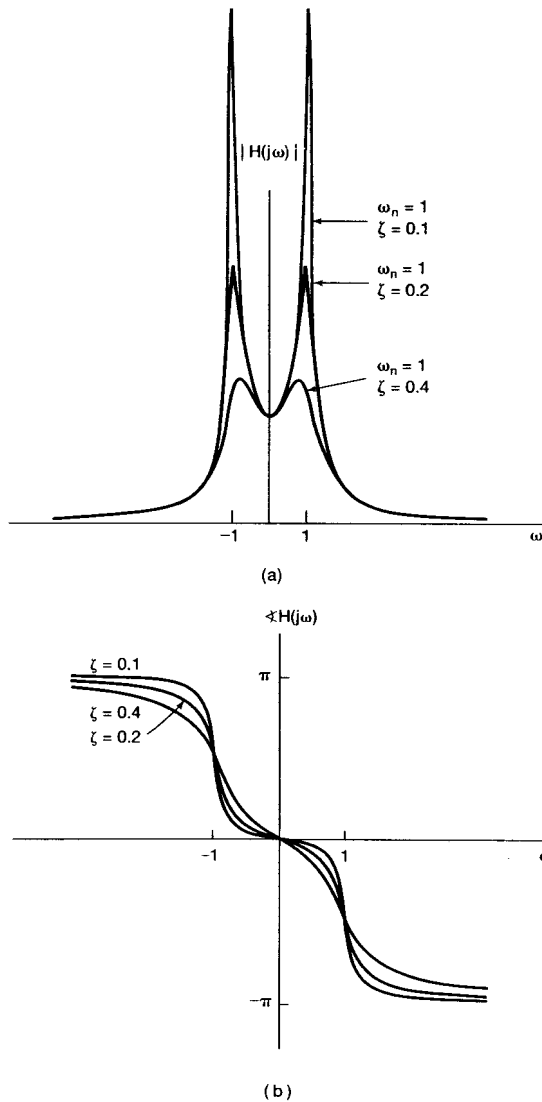


Figure 9.20 (a) Magnitude and (b) phase of the frequency response for a second-order system with $0 < \zeta < 1$.

Thus, for $0 < \zeta < 1$, the second-order system is a nonideal bandpass filter, with the parameter ζ controlling the sharpness and width of the peak in the frequency response. In particular, from the geometry in Figure 9.19(d), we see that the length of the pole vector from the second-quadrant pole increases by a factor of $\sqrt{2}$ from its minimum at $\omega = \omega_n \sqrt{1 - \zeta^2}$ when ω increases or decreases from this value by $\zeta \omega_n$. Consequently, for small ζ , and neglecting the effect of the distant third-quadrant pole, $|H(j\omega)|$ is within a factor of $\sqrt{2}$ of its peak value over the frequency range

$$\omega_n \sqrt{1 - \zeta^2} - \zeta \omega_n < \omega < \omega_n \sqrt{1 - \zeta^2} + \zeta \omega_n.$$

If we define the relative bandwidth B as the length of this frequency interval divided by the undamped natural frequency ω_n , we see that

$$B = 2\zeta.$$

Thus, the closer ζ is to zero, the sharper and narrower the peak in the frequency response is. Note also that B is the reciprocal of the quality measure Q for second-order systems defined in Section 6.5.2. Thus, as the quality increases, the relative bandwidth decreases and the filter becomes increasingly frequency selective.

An analogous picture can be developed for $\angle H(\omega)$, which is plotted in Figure 9.20(b) for $\omega_n = 1$ and several values of ζ . As can be seen from Figure 9.19(d), the angle of the second-quadrant pole vector changes from $-\pi/4$ to 0 to $\pi/4$ as ω changes from $\omega_n \sqrt{1 - \zeta^2} - \zeta \omega_n$ to $\omega_n \sqrt{1 - \zeta^2}$ to $\omega_n \sqrt{1 - \zeta^2} + \zeta \omega_n$. For small values of ζ , the angle for the third-quadrant pole changes very little over this frequency interval, resulting in a rapid change in $\angle H(j\omega)$ of approximately $\pi/2$ over the interval, as captured in the figure.

Varying ω_n with ζ fixed only changes the frequency scale in the preceding discussion—i.e., $|H(\omega)|$ and $\angle H(\omega)$ depend only on ω/ω_n . From Figure 9.19(c), we also can readily determine how the poles and system characteristics change as we vary ζ , keeping ω_n constant. Since $\cos \theta = \zeta$, the poles move along a semicircle with fixed radius ω_n . For $\zeta = 0$, the two poles are on the imaginary axis. Correspondingly, in the time domain, the impulse response is sinusoidal with no damping. As ζ increases from 0 to 1, the two poles remain complex and move into the left-half plane, and the vectors from the origin to the poles maintain a constant overall magnitude ω_n . As the real part of the poles becomes more negative, the associated time response will decay more quickly as $t \rightarrow \infty$. Also, as we have seen, as ζ increases from 0 toward 1, the relative bandwidth of the frequency response increases, and the frequency response becomes less sharp and less frequency selective.

9.4.3 All-Pass Systems

As a final illustration of the geometric evaluation of the frequency response, let us consider a system for which the Laplace transform of the impulse response has the pole-zero plot shown in Figure 9.21(a). From this figure, it is evident that for any point along the $j\omega$ -axis, the pole and zero vectors have equal length, and consequently, the magnitude of the frequency response is constant and independent of frequency. Such a system is com-

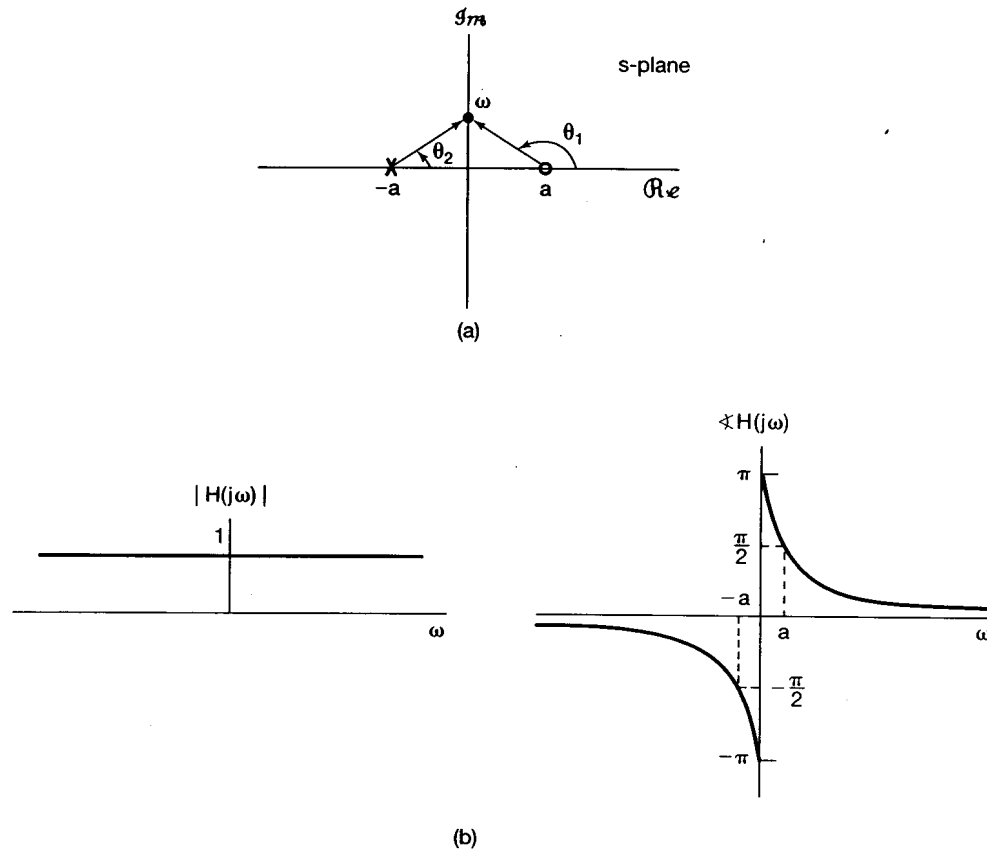


Figure 9.21 (a) Pole-zero plot for an all-pass system; (b) magnitude and phase of an all-pass frequency response.

monly referred to as an *all-pass system*, since it passes all frequencies with equal gain (or attenuation). The phase of the frequency response is $\theta_1 - \theta_2$, or, since $\theta_1 = \pi - \theta_2$,

$$\angle H(j\omega) = \pi - 2\theta_2. \quad (9.80)$$

From Figure 9.21(a), $\theta_2 = \tan^{-1}(\omega/a)$, and thus,

$$\angle H(j\omega) = \pi - 2 \tan^{-1} \left(\frac{\omega}{a} \right). \quad (9.81)$$

The magnitude and phase of $H(j\omega)$ are illustrated in Figure 9.21(b).

9.5 PROPERTIES OF THE LAPLACE TRANSFORM

In exploiting the Fourier transform, we relied heavily on the set of properties developed in Section 4.3. In the current section, we consider the corresponding set of properties for

the Laplace transform. The derivations of many of these results are analogous to those of the corresponding properties for the Fourier transform. Consequently, we will not present the derivations in detail, some of which are left as exercises at the end of the chapter. (See Problems 9.52–9.54.)

9.5.1 Linearity of the Laplace Transform

If

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \quad \text{with a region of convergence that will be denoted as } R_1$$

and

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \quad \text{with a region of convergence that will be denoted as } R_2,$$

then

$$\boxed{ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{L}} aX_1(s) + bX_2(s), \text{ with ROC containing } R_1 \cap R_2.} \quad (9.82)$$

As indicated, the region of convergence of $X(s)$ is at least the intersection of R_1 and R_2 , which could be empty, in which case $X(s)$ has no region of convergence—i.e., $x(t)$ has no Laplace transform. For example, for $x(t)$ as in eq. (9.47) of Example 9.7, with $b > 0$ the ROC for $X(s)$ is the intersection of the ROCs for the two terms in the sum. If $b < 0$, there are no common points in R_1 and R_2 ; that is, the intersection is empty, and thus, $x(t)$ has no Laplace transform. The ROC can also be larger than the intersection. As a simple example, for $x_1(t) = x_2(t)$ and $a = -b$ in eq. (9.82), $x(t) = 0$, and thus, $X(s) = 0$. The ROC of $X(s)$ is then the entire s -plane.

The ROC associated with a linear combination of terms can always be constructed by using the properties of the ROC developed in Section 9.2. Specifically, from the intersection of the ROCs for the individual terms (assuming that it is not empty), we can find a line or strip that is in the ROC of the linear combination. We then extend this to the right ($\Re\{s\}$ increasing) and to the left ($\Re\{s\}$ decreasing) to the nearest poles (which may be at infinity).

Example 9.13

In this example, we illustrate the fact that the ROC for the Laplace transform of a linear combination of signals can sometimes extend beyond the intersection of the ROCs for the individual terms. Consider

$$x(t) = x_1(t) - x_2(t), \quad (9.83)$$

where the Laplace transforms of $x_1(t)$ and $x_2(t)$ are, respectively,

$$X_1(s) = \frac{1}{s+1}, \quad \Re\{s\} > -1, \quad (9.84)$$

and

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \quad \Re\{s\} > -1. \quad (9.85)$$

The pole-zero plot, including the ROCs for $X_1(s)$ and $X_2(s)$, is shown in Figures 9.22(a) and (b). From eq. (9.82),

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}. \quad (9.86)$$

Thus, in the linear combination of $x_1(t)$ and $x_2(t)$, the pole at $s = -1$ is canceled by a zero at $s = -1$. The pole-zero plot for $X(s) = X_1(s) - X_2(s)$ is shown in Figure 9.22(c). The intersection of the ROCs for $X_1(s)$ and $X_2(s)$ is $\Re\{s\} > -1$. However, since the ROC is always bounded by a pole or infinity, for this example the ROC for $X(s)$ can be extended to the left to be bounded by the pole at $s = -2$, as a result of the pole-zero cancellation at $s = -1$.

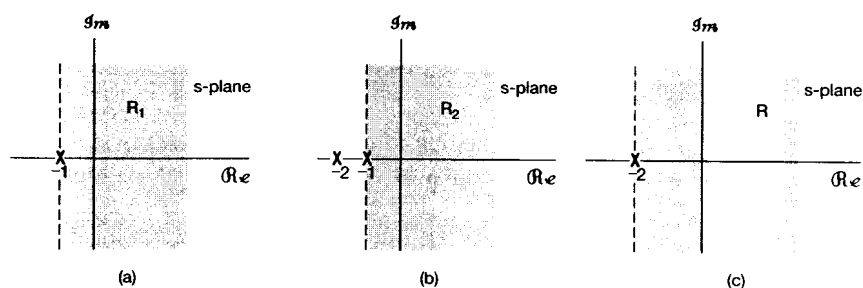


Figure 9.22 Pole-zero plots and ROCs for Example 9.13: (a) $X_1(s)$; (b) $X_2(s)$; (c) $X_1(s) - X_2(s)$. The ROC for $X_1(s) - X_2(s)$ includes the intersection of R_1 and R_2 , which can then be extended to be bounded by the pole at $s = -2$.

9.5.2 Time Shifting

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \quad \text{with ROC} = R,$$

then

$$\boxed{x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s), \quad \text{with ROC} = R.} \quad (9.87)$$