11.4.3 The Nyquist Criterion for Discrete-Time LTI Feedback Systems

As in the continuous-time case, the Nyquist stability criterion for discrete-time systems is based on the fact that the difference in the number of poles and zeros inside a contour, for a rational function, can be determined by examining a plot of the value of the function along the contour. The difference between the continuous-time and discrete-time cases is the choice of the contour. For the discrete-time case, stability of the closed-loop feedback system requires that no zeros of

$$R(z) = \frac{1}{K} + G(z)H(z) \tag{11.94}$$

lie outside the unit circle.

Recall that the encirclement property relates to poles and zeros *inside* any specified contour. On the other hand, in examining the stability of a discrete-time system, we are concerned with the zeros of $R(z)$ *outside* the unit circle. Therefore, in order to make use of the encirclement property, we first make a simple modification. Let us consider the rational function

$$\hat{R}(z) = R\left(\frac{1}{z}\right) \tag{11.95}$$

obtained by replacing z by its reciprocal. As seen in Problem 10.43, if z_0 is a zero (pole) of $R(z)$, then $1/z_0$ is a zero (pole) of $\hat{R}(z)$. Since $1/|z_0|$ is less than 1 if $|z_0| > 1$, any zero or pole of $R(z)$ *outside* the unit circle corresponds to a zero or pole of $\hat{R}(z)$ *inside* the unit circle.

From the basic encirclement property, we know that as z traverses the unit circle in a clockwise direction, the net number of clockwise encirclements of the origin by $\hat{R}(z)$ equals the difference between the number of its zeros and poles inside the unit circle. However, from the previous paragraph, this equals the difference between the number of zeros and poles of $R(z)$ *outside* the unit circle. Furthermore, on the unit circle, $z = e^{j\omega}$ and $1/z = e^{-j\omega}$. Therefore,

$$\hat{R}(e^{j\omega}) = R(e^{-j\omega}). \tag{11.96}$$

From this, we see that evaluating $\hat{R}(z)$ as z traverses the unit circle in the clockwise direction is identical to evaluating $R(z)$ as z traverses the unit circle in the *counterclockwise* direction. In sum, then,

The number of clockwise encirclements of the origin by the plot of $R(e^{j\omega})$ as the unit circle is traversed in the counterclockwise direction (e.g., as ω increases from 0 to 2π)	=	The number of zeros of $R(z)$ outside the unit circle minus the number of poles of $R(z)$ outside the unit circle.	(11.97)

Much as in the continuous-time case, counting the encirclements of the origin by $R(e^{j\omega})$ is equivalent to counting the number of encirclements of the point $-1/K$ by the plot of $G(e^{j\omega})H(e^{j\omega})$, again referred to as the Nyquist plot, which is graphed as ω varies from 0 to 2π. Also, the poles of $R(z)$ are precisely the poles of $G(z)H(z)$, and the zeros of $R(z)$ are the closed-loop poles. Therefore, the encirclement property stated in the preceding paragraph implies that the net number of clockwise encirclements by the Nyquist plot of the point $-1/K$ equals the number of closed-loop poles outside the unit circle minus the number of poles of $G(z)H(z)$ outside the unit circle. In order that the closed-loop system be stable, we require no closed-loop poles outside the unit circle. This yields the ***discrete-time Nyquist stability criterion***:

Discrete-Time Nyquist Stability Criterion: For the closed-loop system to be stable, the net number of *clockwise* encirclements of the point $-1/K$ by the Nyquist plot of $G(e^{j\omega})H(e^{j\omega})$ as ω varies from 0 to 2π must equal *minus* the number of poles of $G(z)H(z)$ that lie outside the unit circle. Equivalently, the net number of *counterclockwise* encirclements of the point $-1/K$ by the Nyquist plot of $G(e^{j\omega})H(e^{j\omega})$ as ω varies from 0 to 2π must *equal* the number of poles of $G(z)H(z)$ outside the unit circle.

Example 11.8

Let

$$G(z)H(z) = \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{1}{z(z + \frac{1}{2})}. \tag{11.98}$$

The Nyquist plot of this curve is shown in Figure 11.24. Since $G(z)H(z)$ has no poles outside the unit circle, for the stability of the closed-loop system there must be no encirclements of the point $-1/K$. From the figure, we see that this will be the case either if $-1/K < -1$ or if $-1/K > 2$. Thus, the system is stable for $-1/2 < K < 1$.

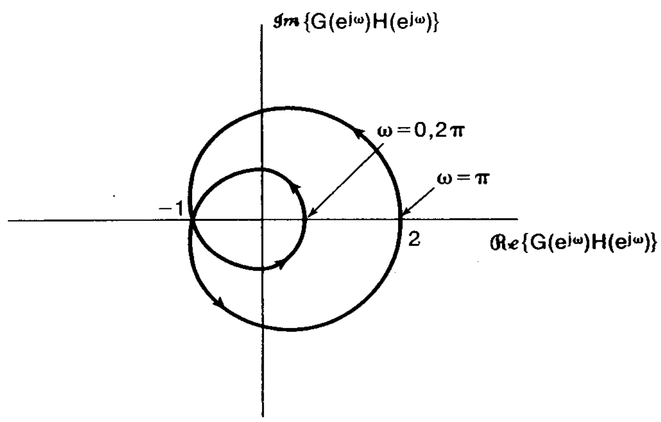


Figure 11.24 Nyquist plot for Example 11.8. The arrow on the curve indicates the direction in which the curve is traversed as ω increases from 0 to 2π.

Just as in continuous time, if the forward and feedback paths are stable, then the Nyquist plot can be obtained from the frequency responses $H(e^{j\omega})$ and $G(e^{j\omega})$ of these systems. If the forward and feedback paths are unstable, then the frequency responses are not defined. Nevertheless, the *function* $G(z)H(z)$ can still be evaluated on the unit circle, and the Nyquist stability criterion can be applied.

As we have seen in this section, the Nyquist stability criterion provides a useful method for determining the range of values of the gain K for which a continuous-time or discrete-time feedback is stable (or unstable). This criterion and the root-locus method are extremely important tools in the design and implementation of feedback systems, and each has its own uses and limitations. For example, the Nyquist criterion can be applied to nonrational system functions, whereas the root-locus method cannot. On the other hand, root-locus plots allow us to examine not only stability, but also other characteristics of the closed-loop system response, such as damping, oscillation frequency, and so on, which are readily identifiable from the location of the poles of the closed-loop system. In the next section, we introduce an additional tool for the analysis of feedback systems that highlights another important characteristic of closed-loop system behavior.

11.5 GAIN AND PHASE MARGINS

In this section, we introduce and examine the concept of the *margin of stability* in a feedback system. It is often of interest not only to know *whether* a feedback system is stable, but also to determine how much the gain in the system can be perturbed and how much additional phase shift can be added to the system before it becomes unstable. Information such as this is important because in many applications the forward and feedback system functions are known only approximately or may change slightly during operation because of wear, the effect of high temperatures on components, or similar influences.

As an example, consider the telescope-pointing system described in Section 11.0 and illustrated in Figures 11.1(c) and (d). This system consists of a motor, a potentiometer converting the shaft angle to a voltage, and an amplifier that is used to amplify the voltage representing the difference between the desired and the actual shaft angles. Assuming that we have obtained approximate descriptions of each of these components, we can set the amplifier gain so that the system will be stable if these descriptions are accurate. However, the amplifier gain and the constant of proportionality that describes the angle-voltage characteristic of the potentiometer are never known exactly, and therefore, the actual gain in the feedback system may differ from the nominal value assumed in designing the system. Furthermore, the damping characteristics of the motor cannot be determined with absolute precision, and thus, the actual time constant of the motor response may differ from the approximate value in the specification of the system. For example, if the actual motor time constant is larger than the nominal value used in the design, the motor will respond more sluggishly than anticipated, thereby producing an effective time delay in the feedback system. As we have discussed in earlier chapters, and as we will again in Example 11.11, time delays have the effect of increasing the negative phase in the frequency response of a system, and this phase shift can have a destabilizing influence on the system. Because of the possible presence of gain and phase errors such as those we have just described, it is clearly desirable to set the amplifier gain so that there is some margin for error—that is, so that the actual system will remain stable even if it differs somewhat from the approximate model used in the design process.

In this section, we introduce one method for quantifying the margin of stability in a feedback system. To do this, we consider a closed-loop system, depicted in Figure 11.25, that has been designed to be stable based on nominal values for the forward- and feedback-path system functions. For our discussion here, we let $H(s)$ and $G(s)$ denote these nominal values. Also, since the basic concepts are identical for both continuous-time and discrete-time systems, we will again focus our development on the continuous-time case, and at the end of the section we illustrate the application of these ideas to a discrete-time example.

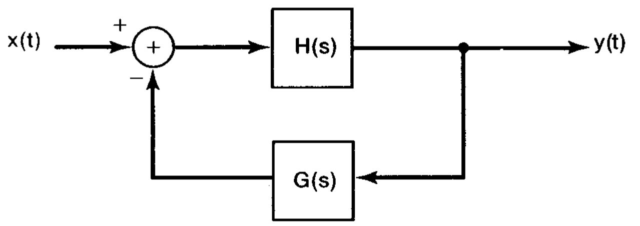


Figure 11.25 Typical feedback system designed to be stable, assuming nominal descriptions for $H(s)$ and $G(s)$.

To assess the margin of stability in our feedback system, suppose that the actual system is as depicted in Figure 11.26, where we have allowed for the possibility of a gain K and phase shift ϕ in the feedback path. In the nominal system K is unity and ϕ is zero, but in the actual system either or both may have a different value. Therefore, it is of interest to know how much variation can be tolerated in these quantities without losing closed-loop system stability. In particular, the *gain margin* of the feedback system is defined as the minimum amount of additional gain K, with $\phi = 0$, that is required so that the closed-loop system becomes unstable. Similarly, the *phase margin* is the additional amount of phase shift, with $K = 1$, that is required for the system to be unstable. By convention, the phase margin is expressed as a positive quantity; that is, it equals the magnitude of the additional negative phase shift at which the feedback system becomes unstable.

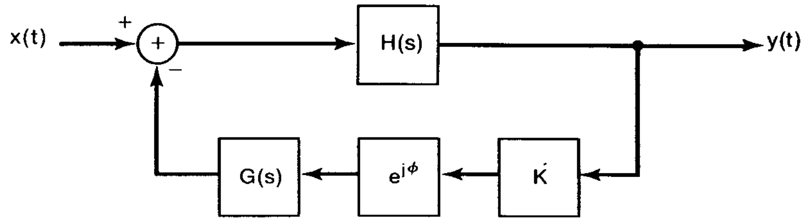


Figure 11.26 Feedback system containing possible gain and phase deviations from the nominal description depicted in Figure 11.25.

Since the closed-loop system of Figure 11.25 is stable, the system of Figure 11.26 can become unstable if, as K and ϕ are varied, at least one pole of the closed-loop system crosses the $j\omega$-axis. If a pole of the closed-loop system is on the $j\omega$-axis at, say, $\omega = \omega_0$, then at this frequency

$$1 + Ke^{-j\phi}G(j\omega_0)H(j\omega_0) = 0, \tag{11.99}$$

or

$$Ke^{-j\phi}G(j\omega_0)H(j\omega_0) = -1. \tag{11.100}$$

Note that with $K = 1$ and $\phi = 0$, by our assumption of stability for the nominal feedback system of Figure 11.25, there is no value of ω_0 for which eq. (11.100) is satisfied. The gain margin of this system is the minimum value of $K > 1$ for which eq. (11.100) has a solution for *some* ω_0 with $\phi = 0$. That is, the gain margin is the smallest value of K for which the equation

$$KG(j\omega_0)H(j\omega_0) = -1 \tag{11.101}$$

has a solution ω_0. Similarly, the phase margin is the minimum value of ϕ for which eq. (11.100) has a solution for some ω_0 when $K = 1$. In other words, the phase margin is the smallest value of $\phi > 0$ for which the equation

$$e^{-j\phi}G(j\omega_0)H(j\omega_0) = -1 \tag{11.102}$$

has a solution.

To illustrate the calculation and graphical interpretation of gain and phase margins, we consider the following example.

Example 11.9

Let

$$G(s)H(s) = \frac{4(1 + \frac{1}{2}s)}{s(1 + 2s)(1 + 0.05s + (0.125s)^2)}. \tag{11.103}$$

The Bode plot for this example is shown in Figure 11.27. Note that, as discussed in Problem 6.31, the factor of $1/j\omega$ in $G(j\omega)H(j\omega)$ contributes $-90°$ ($-\pi/2$ radians) of phase shift and a 20-dB-per-decade increase in $|G(j\omega)H(j\omega)|$. To determine the gain margin, we observe that, with $\phi = 0$, the only frequency at which eq. (11.101) can be satisfied is that for which $\sphericalangle G(j\omega_0)H(j\omega_0) = -\pi$. At this frequency, the gain margin in decibels can be identified by inspection of Figure 11.27. We first examine Figure 11.27(b) to determine the frequency ω_1 at which the angle curve crosses the line $-\pi$ radians. Locating the point at this same frequency in Figure 11.27(a) provides us with the value of $|G(j\omega_1)H(j\omega_1)|$. For eq. (11.101) to be satisfied for $\omega_0 = \omega_1$, K must equal $1/|G(j\omega_1)H(j\omega_1)|$. This value is the gain margin. As illustrated in Figure 11.27(a), the gain margin expressed in decibels can be identified as the amount the log-magnitude curve would have to be shifted up so that the curve intersects the 0-dB line at the frequency ω_1.

In a similar fashion, we can determine the phase margin. Note first that the only frequency at which eq. (11.102) can be satisfied is that for which $|G(j\omega_0)H(j\omega_0)| = 1$, or equivalently, $20 \log_{10} |G(j\omega_0)H(j\omega_0)| = 0$. To determine the phase margin, we first find the frequency ω_2 in Figure 11.27(a) at which the log-magnitude curve crosses the 0-dB line. Locating the point at this same frequency in Figure 11.27(b) then provides us with the value of $\sphericalangle G(j\omega_2)H(j\omega_2)$. For eq. (11.102) to be satisfied for $\omega_0 = \omega_2$, the angle of the left-hand side of this equation must be $-\pi$. The value of ϕ for which this is true is the phase margin. As illustrated in Figure 11.27(b), the phase margin can be identified as the amount the angle curve would have to be lowered so that the curve intersects the line $-\pi$ at the frequency ω_2.

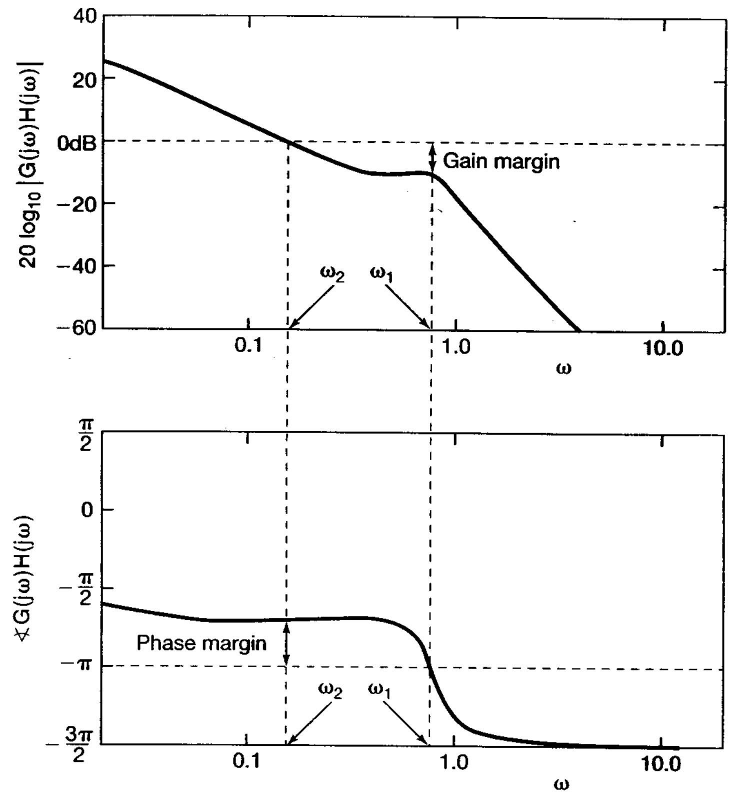


Figure 11.27 Use of Bode plots to calculate gain and phase margins for the system of Example 11.9.

In determining gain and phase margins, it is not always of interest to identify explicitly the *frequency* at which the poles will cross the $j\omega$-axis. As an alternative, we can identify the gain and phase margins from a *log magnitude-phase diagram.* For example, the log magnitude-phase diagram for the system of Figure 11.27 is shown in Figure 11.28. In this figure, we plot $20 \log_{10} |G(j\omega)H(j\omega)|$ versus $\sphericalangle G(j\omega)H(j\omega)$ as ω varies from 0 to $+\infty$. Therefore, because of the conjugate symmetry of $G(j\omega)H(j\omega)$, the plot contains the same information as the Nyquist plot, in which $\mathcal{R}e\{G(j\omega)H(j\omega)\}$ is plotted versus $\mathcal{I}m\{G(j\omega)H(j\omega)\}$ for $-\infty < \omega < \infty$. As we have indicated, the phase margin can be read off by locating the intersection of the log magnitude-phase plot with the 0-dB line. That is, the phase margin is the amount of additional negative phase shift required to shift the log magnitude-phase curve so that it intersects the 0-dB line with exactly 180° (π radians) of phase shift. Similarly, the gain margin is directly obtained from the intersection of the log magnitude-phase curve with the line $-\pi$ radians, and this represents the amount of additional gain needed so that the curve crosses the line $-\pi$ with a magnitude of 0 dB.

The following examples provide several other elementary illustrations of log magnitude-phase diagrams:

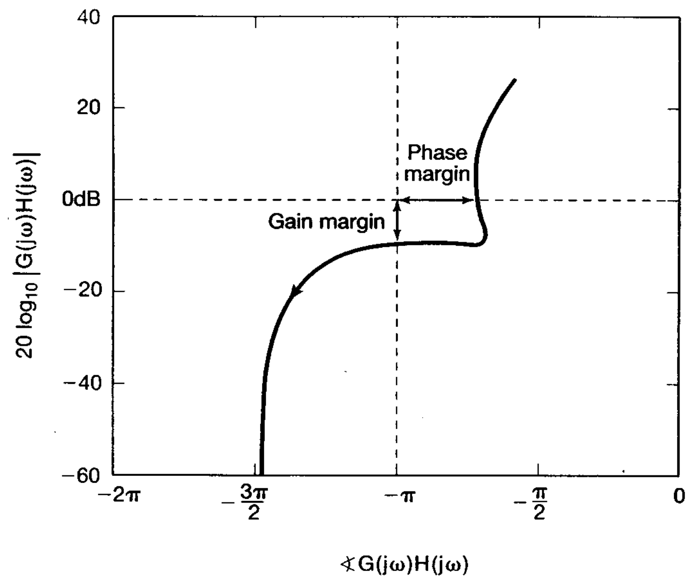


Figure 11.28 Log magnitude-phase plot for the system of Example 11.9.

Example 11.10

Let

$$G(s)H(s) = \frac{1}{\tau s + 1}, \qquad \tau > 0. \tag{11.104}$$

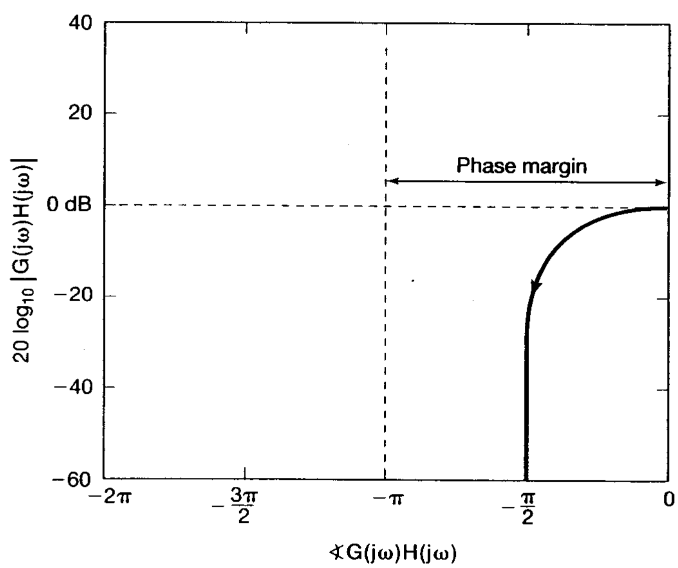


Figure 11.29 Log magnitude-phase plot for the first-order system of Example 11.10.

In this case, we obtain the log magnitude-phase plot depicted in Figure 11.29. This has a phase margin of π, and since the curve does not intersect the line $-\pi$, the system has infinite gain margin (i.e., we can increase the gain as much as we like and maintain stability). This is consistent with the conclusion that we can draw by examining the system illustrated in Figure 11.30(a). In Figure 11.30(b), we have depicted the root locus for this system with $\phi = 0$ and $K > 0$. From the figure, it is evident that the system is stable for any positive value of K. In addition, if $K = 1$ and $\phi = \pi$, so that $e^{j\phi} = -1$, the closed-loop system function for the system of Figure 11.30(a) is $1/\tau s$, which has a pole at $s = 0$, so that the system is unstable.

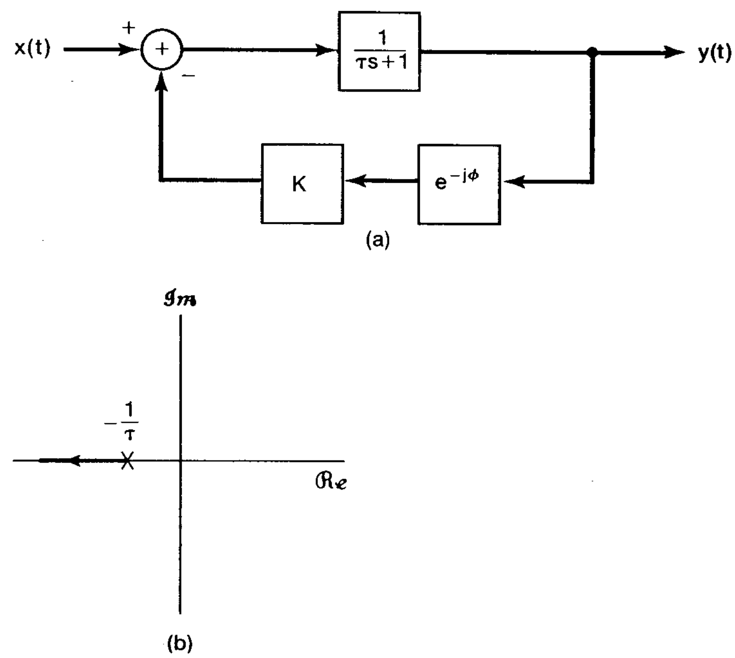


Figure 11.30 (a) First-order feedback system with possible gain and phase variations in the feedback path; (b) root locus for this system with $\phi = 0$, $K > 0$.

Example 11.11

Suppose we now consider the second-order system

$$H(s) = \frac{1}{s^2 + s + 1}, \qquad G(s) = 1. \tag{11.105}$$

The system $H(s)$ has an undamped natural frequency of 1 and a damping ratio of 0.5. The log magnitude-phase plot for this system is illustrated in Figure 11.31. Again we have infinite gain margin, but a phase margin of only $\pi/2$, since it can be shown by a straightforward calculation that $|H(j\omega)| = 1$ for $\omega = 1$, and at this frequency $\measuredangle H(j\omega) = -\pi/2$.

We can now illustrate the type of problem that can be solved using the concepts of gain and phase margins. Suppose that the feedback system specified by eq. (11.105) cannot be realized. Rather, some unavoidable time delay is introduced into the feedback path. That is,

$$G(s) = e^{-s\tau}, \tag{11.106}$$

where τ is the time delay. What we would like to know is how small this delay must be to ensure the stability of the closed-loop system.

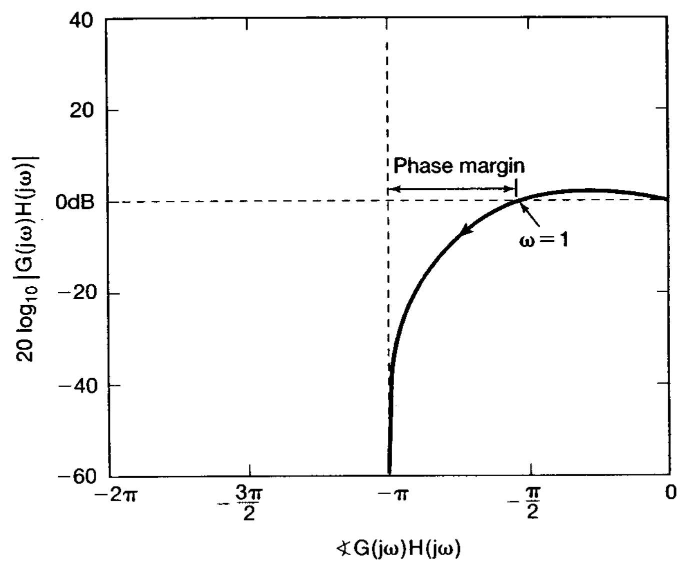


Figure 11.31 Log magnitude-phase plot for the second-order system of Example 11.11.

The first point to note is that

$$|e^{-j\omega\tau}| = 1, \tag{11.107}$$

so the delay does not change the magnitude of $H(j\omega)G(j\omega)$. On the other hand,

$$\sphericalangle e^{-j\omega\tau} = -\omega\tau \text{ radians.} \tag{11.108}$$

Thus, every point on the curve in Figure 11.31 is shifted to the *left*. The amount of the shift is proportional to the value of ω for each point on the log magnitude-phase curve.

From this discussion, we see that instability will occur once the phase margin is reduced to zero, and this will happen when the phase shift introduced by the delay is equal to $-\pi/2$ at $\omega = 1$. That is, the critical value τ^* of the time delay satisfies

$$\sphericalangle e^{-j\tau^*} = -\tau^* = -\frac{\pi}{2}, \tag{11.109}$$

or (assuming that the units of ω are radians/second)

$$\tau^* \approx 1.57 \text{ seconds.} \tag{11.110}$$

Thus, for any time delay $\tau < \tau^*$, the system remains stable.

Example 11.12

Consider again the acoustic feedback system discussed in Section 11.2.6 and Example 11.7. Here, we assume that the system of Figure 11.8 has been designed with $K_1K_2 < 1$, so that the closed-loop system is stable. In this case, the log magnitude-phase plot for

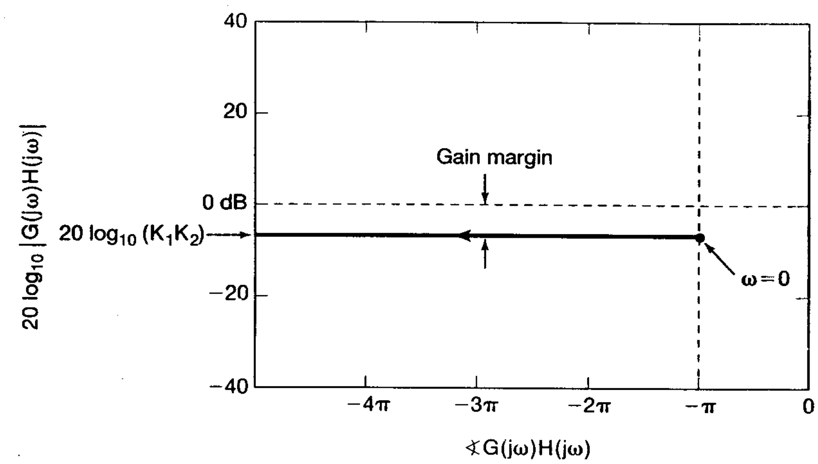


Figure 11.32 Log magnitude-phase plot for Example 11.12.

$G(s)H(s) = K_1K_2e^{-(sT+j\pi)}$ is illustrated in Figure 11.32. From the figure, we see that the system has infinite phase margin and a gain margin in decibels of $-20\log_{10}(K_1K_2)$ (i.e., this is precisely the gain factor that, when multiplied by K_1K_2, equals 1).

As indicated at the start of the section, the definitions of the gain and phase margin are the same for discrete-time feedback systems as for continuous-time systems. Specifically, if we have a stable discrete-time feedback system, the gain margin is the minimum amount of additional gain required in the feedback system such that the closed-loop system becomes unstable. Similarly, the phase margin is the minimum amount of additional negative phase shift required for the feedback system to be unstable. The following example illustrates the graphical calculation of phase and gain margins for a discrete-time feedback system; the procedure is essentially the same as for continuous-time systems.

Example 11.13

In this example, we illustrate the concept of gain and phase margin for the discrete-time feedback system shown in Figure 11.33. Here,

$$G(z)H(z) = \frac{\frac{7\sqrt{2}}{4}z^{-1}}{1 - \frac{7\sqrt{2}}{8}z^{-1} + \frac{49}{64}z^{-2}}, \tag{11.111}$$

and by direct calculation we can check that the feedback system is stable for $K = 1$ and $\phi = 0$. In Figure 11.34, we have displayed the log magnitude-phase diagram for

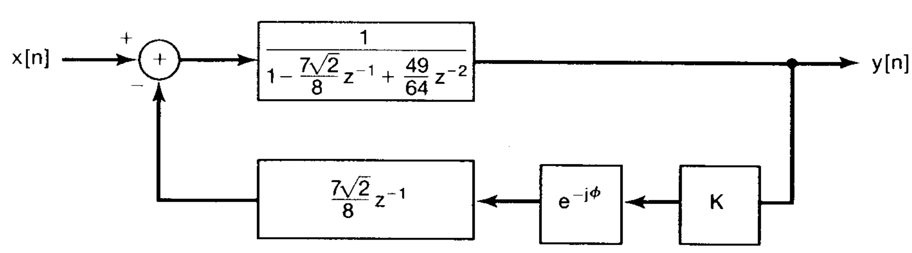


Figure 11.33 Discrete-time feedback system of Example 11.13.

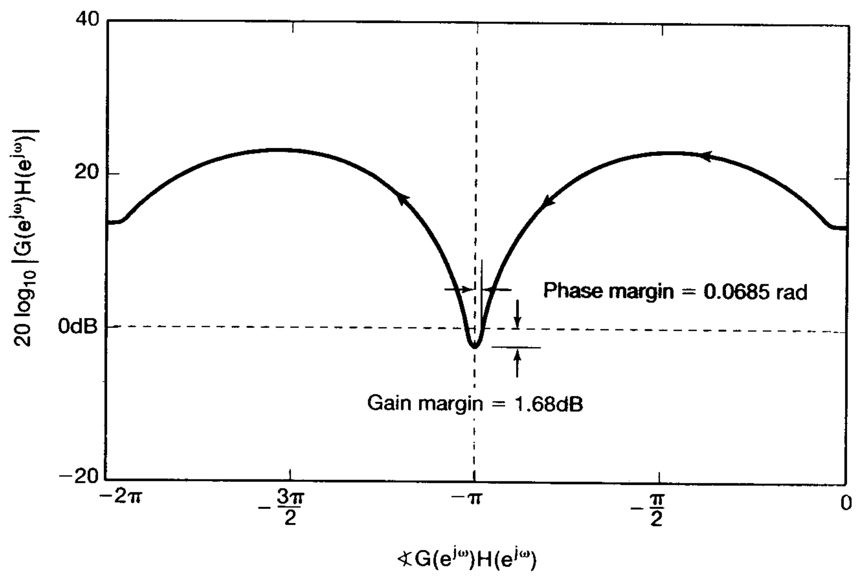


Figure 11.34 Log magnitude-phase diagram for the discrete-time feedback system of Example 11.13.

the system; that is, we have plotted $20 \log_{10}|G(e^{j\omega})H(e^{j\omega})|$ versus $\sphericalangle G(e^{j\omega})H(e^{j\omega})$ as ω varies from 0 to 2π. The system has a gain margin of 1.68 dB and a phase margin of 0.0685 radians (3.93°).

In concluding this section, it should be stressed that the gain margin is the *minimum* value of gain that moves one or more of the closed-loop poles onto the $j\omega$-axis in continuous time or the unit circle in discrete time and, consequently, causes the system to become unstable. It is important to note, however, that this does *not* imply that the system is unstable for *all* values of gain above the value specified by the gain margin. For example, as illustrated in Problem 11.47, as K increases, the root locus may move from the left-half plane into the right-half plane and then cross back into the left-half plane. The gain margin provides us with the information about how much the gain can be increased until the poles *first* reach the $j\omega$-axis, but it tells us nothing about the possibility that the system may again be stable for even larger values of the gain. To obtain such information, we must either refer to the root locus or use the Nyquist stability criterion. (See Problem 11.47.).[4]

11.6 SUMMARY

In this chapter, we have examined a number of the applications and several techniques for the analysis of feedback systems. We have seen how the use of Laplace and z-transforms allows us to analyze these systems algebraically and graphically. In Section 11.2 we indicated several of the applications of feedback, including the design of inverse systems,

[4]For detailed discussions of this point and also of gain and phase margins and log magnitude-phase diagrams in general, see the texts on feedback listed in the bibliography at the end of the book.

the stabilization of unstable systems, and the design of tracking systems. We also saw that feedback can destabilize, as well as stabilize, a system.

In Section 11.3, we described the root-locus method for plotting the poles of the closed-loop system as a function of a gain parameter. Heıe, we found that the geometric evaluation of the phase of a rational Laplace transform or z-transform allowed us to gain a significant amount of insight into the properties of the root locus. These properties often permit us to obtain a reasonably accurate sketch of the root locus without performing complex calculations.

In contrast to the root-locus method, the Nyquist criterion of Section 11.4 is a technique for determining the stability of a feedback system, again as a function of a variable gain, *without* obtaining a detailed description of the location of the closed-loop poles. The Nyquist criterion is applicable to nonrational system functions and thus can be used when all that is available are experimentally determined frequency responses. The same is true of the gain and phase margins described in Section 11.5. These quantities provide a measure of the margin of stability in a feedback system and therefore are of importance to designers in that they allow them to determine how robust a feedback system is to discrepancies between estimates of the forward- and feedback-path system functions and their actual values.

Chapter 11 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

11.1. Consider the interconnection of discrete-time LTI systems shown in Figure P11.1. Express the overall system function for this interconnection in terms of $H_0(z)$, $H_1(z)$, and $G(z)$.

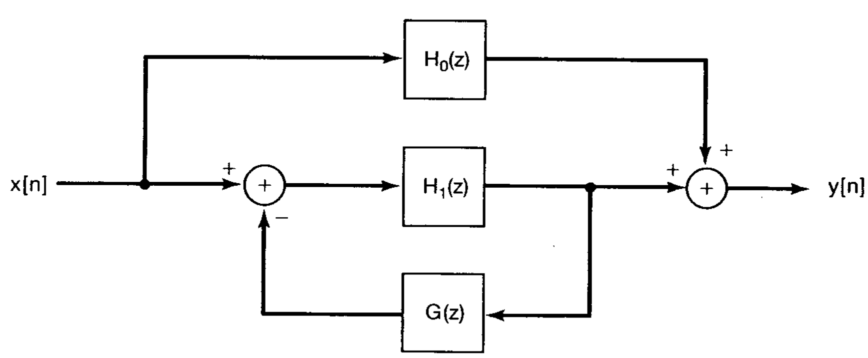


Figure P11.1

11.2. Consider the interconnection of continuous-time LTI systems shown in Figure P11.2. Express the overall system function for this interconnection in terms of $H_1(s)$, $H_2(s)$, $G_1(s)$, and $G_2(s)$.

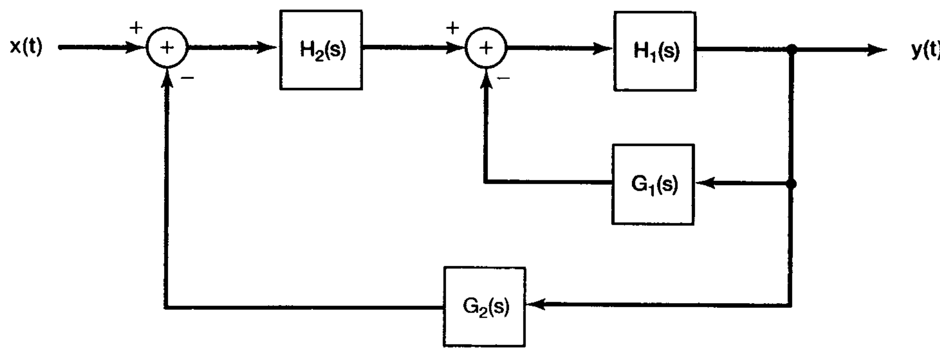


Figure P11.2

11.3. Consider the continuous-time feedback system depicted in Figure 11.3(a) with

$$H(s) = \frac{1}{s-1} \quad \text{and} \quad G(s) = s - b.$$

For what real values of b is the feedback system stable?

11.4. A causal LTI system S with input $x(t)$ and output $y(t)$ is represented by the differential equation

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt}.$$

S is to be implemented using the feedback configuration of Figure 11.3(a) with $H(s) = 1/(s+1)$. Determine $G(s)$.

11.5. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad \text{and} \quad G(z) = 1 - bz^{-1}.$$

For what real values of b is the feedback system stable?

11.6. Consider the discrete-time feedback system depicted in Figure 11.3(b) with

$$H(z) = 1 - z^{-N} \quad \text{and} \quad G(z) = \frac{z^{-1}}{1 - z^{-N}}.$$

Is this system IIR or FIR?

11.7. Suppose the closed-loop poles of a feedback system satisfy

$$\frac{1}{(s+2)(s+3)} = -\frac{1}{K}.$$

Use the root-locus method to determine the values of K for which the feedback system is guaranteed to be stable.

11.8. Suppose the closed-loop poles of a feedback system satisfy

$$\frac{s-1}{(s+1)(s+2)} = -\frac{1}{K}.$$

Use the root-locus method to determine the negative values of K for which the feedback system is guaranteed to be stable.

11.9. Suppose the closed-loop poles of a feedback system satisfy

$$\frac{(s+1)(s+3)}{(s+2)(s+4)} = -\frac{1}{K}.$$

Use the root-locus method to determine whether there are any values of the adjustable gain K for which the system's impulse response has an oscillatory component of the form $e^{-at}\cos(\omega_0 t + \phi)$, where $\omega_0 \neq 0$.

11.10. The root locus corresponding to $G(s)H(s) = -1/K$ is illustrated in Figure P11.10. In this figure, the start ($K = 0$) and end of each branch of the root locus are marked by a '•' symbol. Specify the poles and zeros of $G(s)H(s)$.

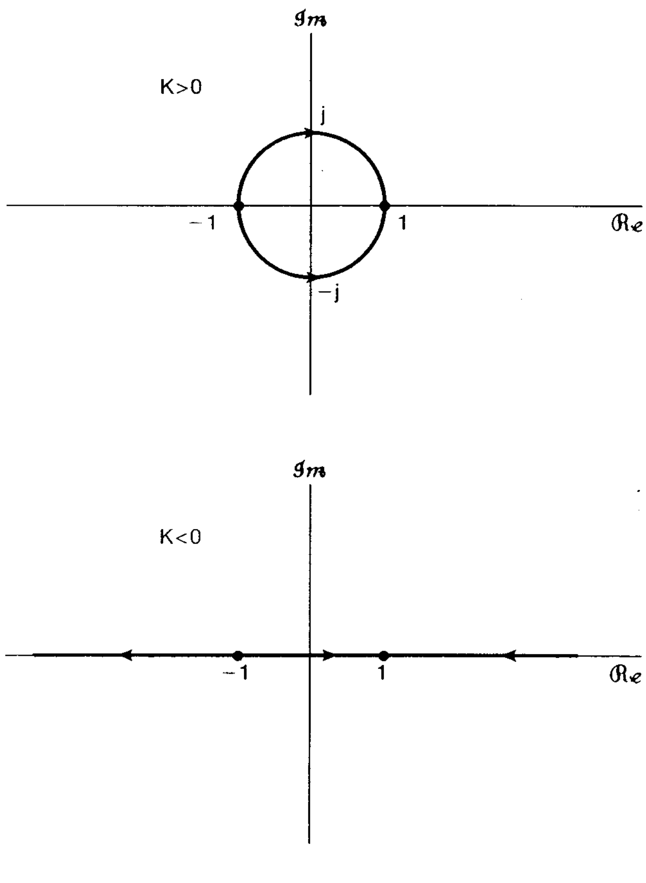


Figure P11.10

11.11. Suppose the closed-loop poles of a discrete-time feedback system satisfy

$$\frac{z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1})} = -\frac{1}{K}.$$

Using the root-locus method, determine the positive values of K for which this system is stable.

11.12. Each of the four locations $z = 1/2$, $z = 1/4$, $z = 0$, and $z = -1/2$ is a single-order pole or zero of $G(z)H(z)$. Furthermore, $G(z)H(z)$ is known to have only two poles. What information can you deduce about the poles and zeros of $G(z)H(z)$ from the fact that for all K, the root locus corresponding to

$$G(z)H(z) = -\frac{1}{K}$$

is on the real axis.

11.13. Consider the block diagram of Figure P11.13 for a discrete-time system. Use the root-locus method to determine the values of K for which the system is guaranteed to be stable.

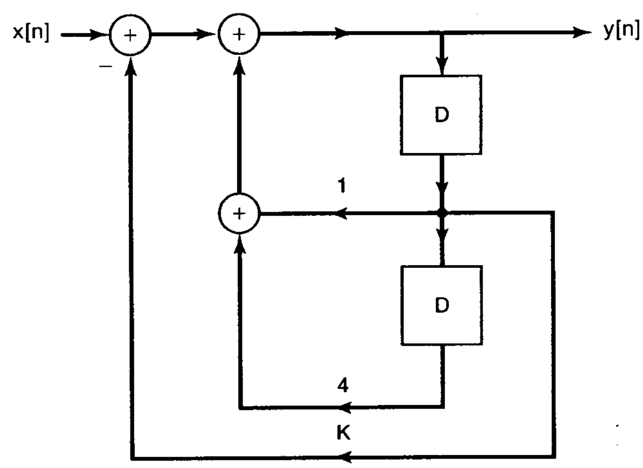


Figure P11.13

11.14. Let C be a closed path that lies on the unit circle in the p-plane and that is traversed in the clockwise direction in order to evaluate $W(p)$. For each of the following expressions for $W(p)$, determine the net number of times the plot of $W(p)$ encircles the origin in a clockwise direction:

(a) $W(p) = \frac{(1-\frac{1}{2}p^{-1})}{(1-\frac{1}{4}p^{-1})}$

(b) $W(p) = \frac{(1-2p^{-1})}{(1-\frac{1}{2}p^{-1})(1-2p^{-1}+4p^{-2})}$

11.15. Consider a continuous-time feedback system whose closed-loop poles satisfy

$$G(s)H(s) = \frac{1}{(s+1)} = -\frac{1}{K}.$$

Use the Nyquist plot and the Nyquist stability criterion to determine the range of values of K for which the closed-loop system is stable. *Hint:* In sketching the Nyquist plot, you may find it useful to sketch the corresponding Bode plot first. It also is helpful to determine the values of ω for which $G(j\omega)H(j\omega)$ is real.

11.16. Consider a continuous-time feedback system whose closed-loop poles satisfy

$$G(s)H(s) = \frac{1}{(s+1)(s/10+1)} = -\frac{1}{K}.$$

Use the Nyquist plot and the Nyquist stability criterion to determine the range of values of K for which the closed-loop system is stable.

11.17. Consider a continuous-time feedback system whose closed-loop poles satisfy

$$G(s)H(s) = \frac{1}{(s+1)^4} = -\frac{1}{K}.$$

Use the Nyquist plot and the Nyquist stability criterion to determine the range of values of K for which the closed-loop system is stable.

11.18. Consider a discrete-time feedback system whose closed-loop poles satisfy

$$G(z)H(z) = z^{-3} = -\frac{1}{K}.$$

Use the Nyquist plot and the Nyquist stability criterion to determine the range of values of K for which the closed-loop system is stable.

11.19. Consider a feedback system, either in continuous-time or discrete-time, and suppose that the Nyquist plot for the system passes through the point $-1/K$. Is the feedback system stable or unstable for this value of the gain? Explain your answer.

11.20. Consider the basic continuous-time feedback system of Figure 11.3(a). Determine the phase and gain margin for the following specification of $H(s)$ and $G(s)$:

$$H(s) = \frac{s+1}{s^2+s+1}, \quad G(s) = 1.$$

BASIC PROBLEMS

11.21. Consider the feedback system of Figure P11.21. Find the closed-loop poles and zeros of this system for the following values of K:

(i) $K = 0.1$
(ii) $K = 1$
(iii) $K = 10$
(iv) $K = 100$

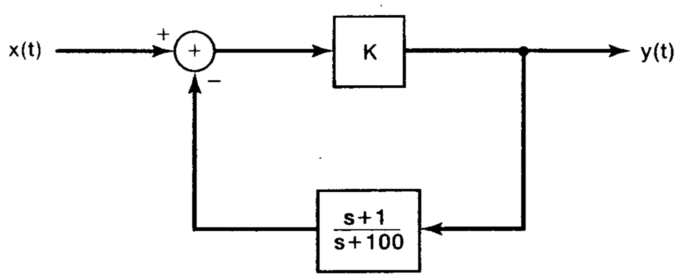


Figure P11.21

11.22. Consider the basic feedback system of Figure 11.3(a). Determine the closed-loop system impulse response for each of the following specifications of the system functions in the forward and feedback paths:

(a) $H(s) = \frac{1}{(s+1)(s+3)}$, $G(s) = 1$

(b) $H(s) = \frac{1}{s+3}$, $G(s) = \frac{1}{s+1}$

(c) $H(s) = \frac{1}{2}$, $G(s) = e^{-s/3}$

11.23. Consider the basic feedback systems of Figure 11.3(b). Determine the closed-loop system impulse response for each of the following specifications of the system functions in the forward and feedback paths:

(a) $H(z) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$, $G(z) = \frac{2}{3} - \frac{1}{6}z^{-1}$

(b) $H(z) = \frac{2}{3} - \frac{1}{6}z^{-1}$, $G(z) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$

11.24. Sketch the root loci for $K > 0$ and $K < 0$ for each of the following:

(a) $G(s)H(s) = \frac{1}{s+1}$

(b) $G(s)H(s) = \frac{1}{(s-1)(s+3)}$

(c) $G(s)H(s) = \frac{1}{s^2+s+1}$

(d) $G(s)H(s) = \frac{s+1}{s^2}$

(e) $G(s)H(s) = \frac{(s+1)^2}{s^3}$

(f) $G(s)H(s) = \frac{s^2+2s+2}{s^2(s-1)}$

(g) $G(s)H(s) = \frac{(s+1)(s-1)}{s(s^2+2s+2)}$

(h) $G(s)H(s) = \frac{(1-s)}{(s+2)(s+3)}$

11.25. Sketch the root loci for $K > 0$ and $K < 0$ for each of the following:

(a) $G(z)H(z) = \frac{z-1}{z^2-\frac{1}{4}}$

(b) $G(z)H(z) = \frac{2}{z^2-\frac{1}{4}}$

(c) $G(z)H(z) = \frac{z^{-1}(1+z^{-1})}{1-\frac{1}{4}z^{-2}}$

(d) $G(z)H(z) = z^{-1} - z^{-2}$

(e) $G(z)H(z)$ is the system function of the causal LTI system described by the difference equation

$$y[n] - 2y[n-1] = x[n-1] - x[n-2].$$

11.26. Consider a feedback system with

$$G(s)H(s) = \frac{(s-a)(s-b)}{s(s+3)(s+6)}.$$

Sketch the root locus for $K > 0$ and $K < 0$ for the following values of a and b:

(a) $a = 1, \quad b = 2$ **(b)** $a = -2, \quad b = 2$ **(c)** $a = -4, \quad b = 2$
(d) $a = -7, \quad b = 2$ **(e)** $a = -1, \quad b = -2$ **(f)** $a = -4, \quad b = 2$
(g) $a = -7, \quad b = -2$ **(h)** $a = -5, \quad b = -4$ **(i)** $a = -7, \quad b = -4$
(j) $a = -7, \quad b = -8$

11.27. Consider a feedback system with

$$H(s) = \frac{s+2}{s^2+2s+4}, \quad G(s) = K.$$

(a) Sketch the root locus for $K > 0$.
(b) Sketch the root locus for $K < 0$.
(c) Find the smallest positive value of K for which the closed-loop impulse response does not exhibit any oscillatory behavior.

11.28. Sketch the Nyquist plot for each of the following specifications of $G(s)H(s)$, and use the continuous-time Nyquist criterion to determine the range of values of K (if any such range exists) for which the closed-loop system is stable. ***Note:*** In sketching the Nyquist plots, you may find it useful to sketch the corresponding Bode plots first. It also is helpful to determine the values of ω for which $G(j\omega)H(j\omega)$ is real.

(a) $G(s)H(s) = \frac{1}{s-1}$ **(b)** $G(s)H(s) = \frac{1}{s^2-1}$
(c) $G(s)H(s) = \frac{1}{(s+1)^2}$ **(d)** $G(s)H(s) = \frac{1}{(s+1)^3}$
(e) $G(s)H(s) = \frac{1-s}{(s+1)^2}$ **(f)** $G(s)H(s) = \frac{s+1}{(s-1)^2}$
(g) $G(s)H(s) = \frac{s+1}{s^2-4}$ **(h)** $G(s)H(s) = \frac{1}{s^2+2s+2}$
(i) $G(s)H(s) = \frac{s+1}{s^2-2s+2}$ **(j)** $G(s)H(s) = \frac{s+1}{(s+100)(s-1)^2}$
(k) $G(s)H(s) = \frac{s^2}{(s+1)^3}$

11.29. Consider the basic continuous-time feedback system of Figure 11.3(a). Sketch the log magnitude-phase diagram, and roughly determine the phase and gain margin, for each of the following choices of $G(s)$ and $H(s)$. You may find it useful to use the straight-line approximations to the Bode plots developed in Chapter 6 to aid you in sketching the log magnitude-phase diagrams. Be careful, however, to take into account how the actual frequency response deviates from its approximation near break frequencies when there are underdamped second-order terms present. (See Section 6.5.2.)

(a) $H(s) = \frac{10s+1}{s^2+s+1}, \ G(s) = 1$
(b) $H(s) = \frac{s/10+1}{s^2+s+1}, \ G(s) = 1$
(c) $H(s) = \frac{1}{(s+1)^2(s+10)}, \ G(s) = 100$
(d) $H(s) = \frac{1}{(s+1)^3}, \ G(s) = \frac{1}{s+1}$
(e) $H(s) = \frac{1-s}{(s+1)(s+10)}, \ G(s) = 1$

(f) $H(s) = \frac{1-s/100}{(s+1)^2}$, $G(s) = \frac{10s+1}{s/10+1}$

(g) $H(s) = \frac{1}{s(s+1)}$, $G(s) = \frac{1}{s+1}$

Note: Your sketch for part (g) should reflect the fact that for this feedback system $|G(j\omega)H(j\omega)| \to \infty$ as $\omega \to 0$; what is the phase of $G(j\omega)H(j\omega)$ for $\omega = 0^+$, i.e., for ω an infinitesimal amount larger than 0?

11.30. Sketch the Nyquist plot for each of the following specifications of $G(z)H(z)$, and use the discrete-time Nyquist criterion to determine the range of values of K (if any such range exists) for which the closed-loop system is stable. [*Note:* In sketching the Nyquist plots, you may find it useful to first sketch the magnitude and phase plots as a function of frequency or at least calculate $|G(e^{j\omega})H(e^{j\omega})|$ and $\sphericalangle G(e^{j\omega})H(e^{j\omega})$ at several points. Also, it is helpful to determine the values of ω for which $G(e^{j\omega})H(e^{j\omega})$ is real.]

(a) $G(z)H(z) = \frac{1}{z-\frac{1}{2}}$ (b) $G(z)H(z) = \frac{1}{z-2}$

(c) $G(z)H(z) = z^{-1}$ (d) $G(z)H(z) = z^{-2}$

(e) $G(z)H(z) = \frac{1}{(z+\frac{1}{2})(z-\frac{3}{2})}$ (f) $G(z)H(z) = \frac{z-\sqrt{3}}{z(z+1/\sqrt{3})}$

(g) $G(z)H(z) = \frac{1}{z^2-z+\frac{1}{3}}$ (h) $G(z)H(z) = \frac{z-\frac{1}{2}}{z(z-2)}$

(i) $G(z)H(z) = \frac{(z+1)^2}{z^3}$

11.31. Consider the basic discrete-time system in Figure 11.3(b). Sketch the log magnitude-phase diagram, and roughly determine the phase and gain margin, for each of the following choices of $G(z)$ and $H(z)$. You may find it useful to determine the values of ω for which either $|G(e^{j\omega})H(e^{j\omega})| = 1$ or $\sphericalangle G(e^{j\omega}) = -\pi$.

(a) $H(z) = z^{-1}$, $G(z) = \frac{1}{2}$

(b) $H(z) = \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$, $G(z) = \frac{1}{2}$

(c) $H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{2}z^{-1})}$, $G(z) = z^{-2}$

(d) $H(z) = \frac{2}{z-2}$, $G(z) = 1$

(e) $H(z) = \frac{1}{z+\frac{1}{2}}$, $G(z) = \frac{1}{z-\frac{3}{2}}$

(f) $H(z) = \frac{1}{z+\frac{1}{2}}$, $G(z) = 1 - \frac{3}{2}z^{-1}$

(g) $H(z) = \frac{\frac{1}{2}}{z^2-z+\frac{1}{3}}$, $G(z) = 1$

(h) $H(z) = \frac{1}{z-1}$, $G(z) = \frac{1}{4}z^{-1}$

(*Note:* Your sketch for part (h) should reflect the fact that, for this feedback system, $G(z)H(z)$ has a pole at $z = 1$; what are the values of $\sphericalangle G(e^{j\omega})H(e^{j\omega})$ for $e^{j\omega}$ just on either side of the point $z = 1$?)

ADVANCED PROBLEMS

11.32. (a) Consider the feedback system of Figure 11.10(b) with

$$H(s) = \frac{N_1(s)}{D_1(s)}, \quad G(s) = \frac{N_2(s)}{D_2(s)}. \tag{P11.32–1}$$

Assume that there is no pole-zero cancellation in the product $G(s)H(s)$. Show that the zeros of the closed-loop system function consist of the zeros of $H(s)$ and the poles of $G(s)$.

(b) Use the result of part (a) together with the appropriate property of the root locus to confirm that, with $K = 0$, the closed-loop system zeros are the zeros of $H(s)$ and the closed-loop poles are the poles of $H(s)$.

(c) While it is usual for $H(s)$ and $G(s)$ in eq. (P11.32–1) to be in reduced form [i.e., the polynomials $N_1(s)$ and $D_1(s)$ have no common factors, and the same is true of $N_2(s)$ and $D_2(s)$], it may happen that $N_1(s)$ and $D_2(s)$ have common factors or $N_2(s)$ and $D_1(s)$ have common factors. To see what occurs when such common factors are present, let $p(s)$ denote the greatest common factor of $N_1(s)$ and $D_2(s)$. That is,

$$\frac{N_1(s)}{p(s)} \quad \text{and} \quad \frac{D_2(s)}{p(s)}$$

are both polynomials and have *no* common factors. Similarly,

$$\frac{N_2(s)}{q(s)} \quad \text{and} \quad \frac{D_1(s)}{q(s)}$$

are polynomials and have no common factors. Show that the closed-loop system function can be written as

$$Q(s) = \frac{p(s)}{q(s)}\left[\frac{\hat{H}(s)}{1 + K\hat{G}(s)\hat{H}(s)}\right], \tag{P11.32–2}$$

where

$$\hat{H}(s) = \frac{N_1(s)/p(s)}{D_1(s)/q(s)}$$

and

$$\hat{G}(s) = \frac{N_2(s)/q(s)}{D_2(s)/p(s)}.$$

Therefore, from eq. (P11.32–2) and part (a), we see that the zeros of $Q(s)$ are the zeros of $p(s)$, the zeros of $\hat{H}(s)$, and the poles of $\hat{G}(s)$, while the poles of $Q(s)$ are the zeros of $q(s)$ and the solutions of

$$1 + K\hat{G}(s)\hat{H}(s) = 0. \tag{P11.32–3}$$

By construction, there is no pole-zero cancellation in the product $\hat{G}(s)\hat{H}(s)$, and thus, we can apply the root-locus method described in Section 11.3 to sketch the locations of the solutions of eq. (P11.32–3) as K is varied.

(d) Use the procedure outlined in part (c) to determine the closed-loop zeros, any closed-loop poles whose locations are independent of K, and the locus of the remaining closed-loop poles for $K > 0$ when

$$H(s) = \frac{s+1}{(s+4)(s+2)}, \quad G(s) = \frac{s+2}{s+1}.$$

(e) Repeat part (d) for

$$H(z) = \frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad G(z) = \frac{z^{-1}}{1+z^{-1}}.$$

(f) Let

$$H(z) = \frac{z^2}{(z-2)(z+2)}, \quad G(z) = \frac{1}{z^2}.$$

(i) Sketch the root locus for $K > 0$ and for $K < 0$.
(ii) Find all the values of K for which the overall system is stable.
(iii) Find the impulse response of the closed-loop system when $K = 4$.

11.33. Consider the feedback system of Figure 11.10(a), and suppose that

$$G(s)H(s) = \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)},$$

where $m > n$.[5] In this case $G(s)H(s)$ has $m - n$ poles at infinity (see Chapter 9), and we can adapt the root-locus rules given in the text by noting that (1) there are m branches of the root locus and (2) for $K = 0$, all branches of the root locus begin at poles of $G(s)H(s)$, $m-n$ of which are at infinity. Furthermore, as $|K| \to \infty$, these branches converge to the m zeros of $G(s)H(s)$, namely, $\beta_1, \beta_2, \ldots, \beta_m$. Use these facts to assist you in sketching the root locus (for $K > 0$ and for $K < 0$) for each of the following:

(a) $G(s)H(s) = s - 1$
(b) $G(s)H(s) = (s+1)(s+2)$
(c) $G(s)H(s) = \frac{(s+1)(s+2)}{s-1}$

11.34. In Section 11.3, we derived a number of properties that can be of value in determining the root locus for a feedback system. In this problem, we develop several additional properties. We derive these properties in terms of continuous-time systems, but, as with all root-locus properties, they hold as well for discrete-time root loci. For our discussion of these properties, we refer to the basic equation satisfied by the closed-loop poles, namely,

$$G(s)H(s) = -\frac{1}{K}, \tag{P11.34–1}$$

[5]Note that for a continuous-time system, the condition $m > n$ implies that the system with system function $G(s)H(s)$ involves differentiation of the input. [In fact, the inverse transform of $G(s)H(s)$ includes singularity functions up to the order $m - n$.] In discrete time, if $G(z)H(z)$, written as a ratio of polynomials in z, has $m > n$, it is necessarily the system function of a noncausal system. [In fact, the inverse transform of $G(z)H(z)$ has a nonzero value at time $n - m < 0$.] Thus, the case considered in this problem is actually of interest only for continuous-time systems.

where

$$G(s)H(s) = \frac{\prod_{k=1}^{m}(s-\beta_k)}{\prod_{k=1}^{n}(s-\alpha_k)} = \frac{\sum_{k=0}^{m} b_k s^k}{\sum_{k=0}^{n} a_k s^k}. \qquad \text{(P11.34–2)}$$

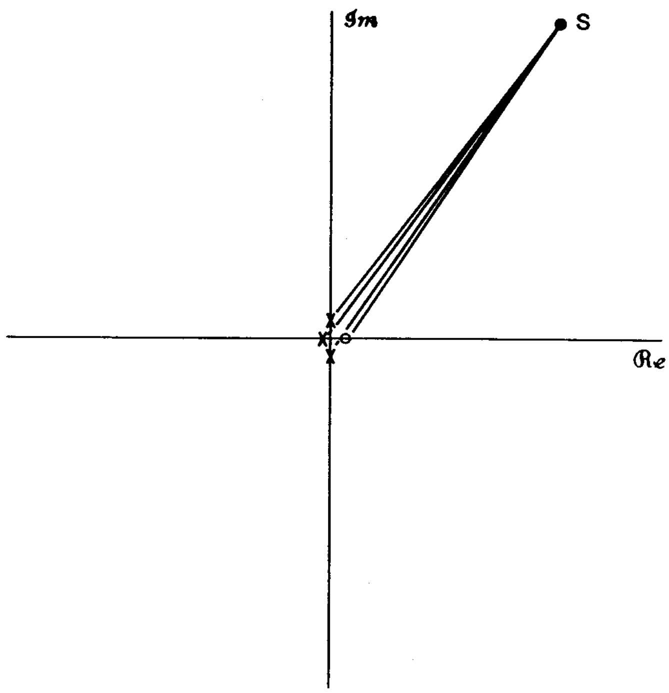


Figure P11.34

Throughout this problem, we assume that $m \leq n$.

(a) From Property 2, we know that $n - m$ branches of the root locus go to zeros of $G(s)H(s)$ located at infinity. In this first part, we demonstrate that it is straightforward to determine the angles at which these branches approach infinity. Specifically, consider searching the remote part of the s-plane [i.e., the region where $|s|$ is extremely large and far from any of the poles and zeros of $G(s)H(s)$]. This region is illustrated in Figure P11.34. Use the geometry of the picture, together with the angle criterion for $K > 0$ and for $K < 0$, to deduce that:

- For $K > 0$, the $n - m$ branches of the root locus that approach infinity do so at the angles

$$\frac{(2k+1)\pi}{n-m}, \qquad k = 0, 1, \ldots, n-m-1.$$

- For $K < 0$, the $n - m$ branches of the root locus that approach infinity do so at the angles

$$\frac{2k\pi}{n-m}, \qquad k = 0, 1, \ldots, n-m-1.$$

Thus, the branches of the root locus that approach infinity do so at specified angles that are arranged symmetrically. For example, for $n - m = 3$ and $K > 0$, we see that the asymptotic angles are $\pi/3$, π, and $5\pi/3$. The result of part (a), together with one additional fact, allows us to draw in the asymptotes for the branches of the root locus that approach infinity. Specifically, all of the $n - m$ asymptotes intersect at a single point on the real axis. This is derived in the next part of the problem.

(b) (i) As a first step, consider a general polynomial equation

$$s^r + f_{r-1}s^{r-1} + \cdots + f_0 = (s - \xi_1)(s - \xi_2)\cdots(s - \xi_r) = 0.$$

Show that

$$f_{r-1} = -\sum_{i=1}^{r} \xi_i.$$

(ii) Perform long division on $1/G(s)H(s)$ to write

$$\frac{1}{G(s)H(s)} = s^{n-m} + \gamma_{n-m-1}s^{n-m-1} + \ldots. \qquad \text{(P11.34–3)}$$

Show that

$$\gamma_{n-m-1} = a_{n-1} - b_{m-1} = \sum_{k=1}^{m} \beta_k - \sum_{k=1}^{n} \alpha_k.$$

[See eq. (P11.34–2).]

(iii) Argue that the solution of eq. (P11.34–1) for *large* s is an approximate solution of the equation

$$s^{n-m} + \gamma_{n-m-1}s^{n-m-1} + \gamma_{n-m-2}s^{n-m-2} + \cdots + \gamma_0 + K = 0.$$

(iv) Use the results of (i)–(iii) to deduce that the sum of the $n - m$ closed-loop poles that approach infinity is asymptotically equal to

$$b_{m-1} - a_{n-1}.$$

Thus, the center of gravity of these $n - m$ poles is

$$\frac{b_{m-1} - a_{n-1}}{n-m},$$

which does not depend on K. Consequently, we have $n - m$ closed-loop poles that approach $|s| = \infty$ at evenly spaced angles and that have a center of gravity that is independent of K. From this, we can deduce that:

The asymptotes of the $n-m$ branches of the root locus that approach infinity intersect at the point

$$\frac{b_{m-1} - a_{n-1}}{n - m} = \frac{\sum_{k=1}^{n} \alpha_k - \sum_{k=1}^{m} \beta_k}{n - m}.$$

This point of intersection of the asymptotes is the same for $K > 0$ and $K < 0$.

(c) Suppose that

$$G(s)H(s) = \frac{1}{(s + 1)(s + 3)(s + 5)}.$$

(i) What are the asymptotic angles for the closed-loop poles that approach infinity for $K > 0$ and for $K < 0$?
(ii) What is the point of intersection of the asymptotes?
(iii) Draw in the asymptotes, and use them to help you sketch the root locus for $K > 0$ and for $K < 0$.

(d) Repeat part (c) for each of the following:
(i) $G(s)H(s) = \frac{s+1}{s(s+2)(s+4)}$
(ii) $G(s)H(s) = \frac{1}{s^4}$
(iii) $G(s)H(s) = \frac{1}{s(s+1)(s+5)(s+6)}$
(iv) $G(s)H(s) = \frac{1}{(s+2)^2(s-1)^2}$
(v) $G(s)H(s) = \frac{s+3}{(s+1)(s^2+2s+2)}$
(vi) $G(s)H(s) = \frac{s+1}{(s+2)^2(s^2+2s+2)}$
(vii) $G(s)H(s) = \frac{s+1}{(s+100)(s-1)(s-2)}$

(e) Use the result of part (a) to explain why the following statement is true: For any continuous-time feedback system, with $G(s)H(s)$ given by eq. (P11.34–2), if $n - m \geq 3$, we can make the closed-loop system unstable by choosing $|K|$ large enough.

(f) Repeat part (c) for the discrete-time feedback system specified by

$$G(z)H(z) = \frac{z^{-3}}{(1 - z^{-1})(1 + \frac{1}{2}z^{-1})}.$$

(g) Explain why the following statement is true: For any discrete-time feedback system with

$$G(z)H(z) = \frac{z^m + b_{m-1}z^{m-1} + \cdots + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_0},$$

if $n > m$, we can make the closed-loop system unstable by choosing $|K|$ large enough.

11.35. (a) Consider again the feedback system of Example 11.2:

$$G(s)H(s) = \frac{s - 1}{(s + 1)(s + 2)}.$$

The root locus for $K < 0$ is plotted in Figure 11.14(b). For some value of K, the closed-loop poles are on the $j\omega$-axis. Determine this value of K and the corresponding locations of the closed-loop poles by examining the real and imaginary parts of the equation

$$G(j\omega)H(j\omega) = -\frac{1}{K},$$

which must be satisfied if the point $s = j\omega$ is on the root locus for any given values of K. Use this result plus the analysis in Example 11.2 to find the full range of values of K (positive and negative) for which the closed-loop system is stable.

(b) Note that the feedback system is unstable for $|K|$ sufficiently large. Explain why this is true in general for continuous-time feedback systems for which $G(s)H(s)$ has a zero in the right-half plane and for discrete-time feedback systems for which $G(z)H(z)$ has a zero outside the unit circle.

11.36. Consider a continuous-time feedback system with

$$G(s)H(s) = \frac{1}{s(s+1)(s+2)}. \tag{P11.36–1}$$

(a) Sketch the root locus for $K > 0$ and for $K < 0$. (*Hint:* The results of Problem 11.34 are useful here.)

(b) If you have sketched the locus correctly, you will see that for $K > 0$, two branches of the root locus cross the $j\omega$-axis, passing from the left-half plane into the right-half plane. Consequently, we can conclude that the closed-loop system is stable for $0 < K < K_0$, where K_0 is the value of the gain for which the two branches of the root locus intersect the $j\omega$-axis. Note that the sketch of the root locus does not by itself tell us what the value of K_0 is or the exact point on the $j\omega$-axis where the branches cross. As in Problem 11.35, determine K_0 by solving the pair of equations obtained as the real and imaginary parts of

$$G(j\omega)H(j\omega) = -\frac{1}{K_0}. \tag{P11.36–2}$$

Determine the corresponding two values of ω (which are the negatives of each other, since poles occur in complex-conjugate pairs).

From your root-locus sketches in part (a), note that there is a segment of the real axis between two poles which is on the root locus for $K > 0$, and a different segment is on the locus for $K < 0$. In both cases, the root locus breaks off from the real axis at some point. In the next part of this problem, we illustrate how one can calculate these breakaway points.

(c) Consider the equation denoting the closed-loop poles:

$$G(s)H(s) = -\frac{1}{K}. \tag{P11.36–3}$$

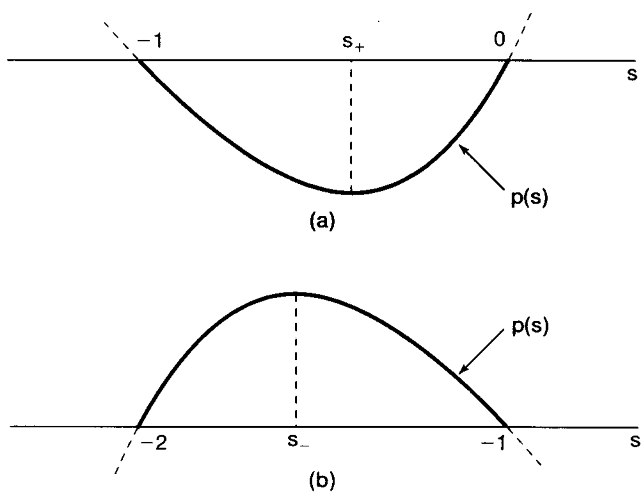


Figure P11.36

Using eq. (P11.36–1), show that an equivalent equation for the closed loop poles is

$$p(s) = s^3 + 3s^2 + 2s = -K. \qquad \text{(P11.36–4)}$$

Consider the segment of the real axis between 0 and -1. This segment is on the root locus for $K \geq 0$. For $K = 0$, two branches of the locus begin at 0 and -1 and approach each other as K is increased.

(i) Use the facts stated, together with eq. (P11.36–4), to explain why the function $p(s)$ has the form shown in Figure P11.36(a) for $-1 \leq s \leq 0$ and why the point s_+ where the minimum occurs is the breakaway point (i.e., it is the point where the two branches of the $K > 0$ locus break from the segment of the real axis between -1 and 0).

Similarly, consider the root locus for $K < 0$ and, more specifically, the segment of the real axis between -1 and -2 that is part of this locus. For $K = 0$, two branches of the root locus begin at -1 and -2, and as K is decreased, these poles approach each other.

(ii) In an analogous fashion to that used in part (i), explain why the function $p(s)$ has the form shown in Figure P11.36(b) and why the point s_- where the maximum occurs is the breakaway point for $K < 0$.

Thus, the breakaway points correspond to the the maxima and minima of $p(s)$ as s ranges over the negative real line.

(iii) The points at which $p(s)$ has a maximum or minimum are the solutions of the equation

$$\frac{dp(s)}{ds} = 0.$$

Use this fact to find the breakaway points s_+ and s_-, and then use eq. (P11.36–4) to find the gains at which these points are closed-loop poles.

In addition to the method illustrated in part (c), there are other, partially analytical, partially graphical methods for determining breakaway points. It is also possible to use a procedure similar to the one just illustrated in part (c) to find the "break-in" points, where two branches of the root locus merge onto the real axis. These methods plus the one illustrated are described in advanced texts such as those listed in the bibliography at the end of the book.

11.37. One issue that must always be taken into account by the system designer is the possible effect of unmodeled aspects of the system one is attempting to stabilize or modify through feedback. In this problem, we provide an illustration of why this is the case. Consider a continuous-time feedback system, and suppose that

$$H(s) = \frac{1}{(s+10)(s-2)} \qquad \text{(P11.37–1)}$$

and

$$G(s) = K. \qquad \text{(P11.37–2)}$$

(a) Use root-locus techniques to show that the closed-loop system will be stable if K is chosen large enough.

(b) Suppose that the system we are trying to stabilize by feedback actually has a system function

$$H(s) = \frac{1}{(s+10)(s-2)(10^{-3}s+1)}. \qquad \text{(P11.37–3)}$$

The added factor can be thought of as representing a first-order system in cascade with the system of eq. (P11.37–1). Note that the time constant of the added first order system is extremely small and thus will appear to have a step response that is almost instantaneous. For this reason, one often neglects such factors in order to obtain simpler and more tractable models that capture all of the important characteristics of the system. However, one must still keep these neglected dynamics in mind in obtaining a useful feedback design. To see why this is the case, show that if $G(s)$ is given by eq. (P11.37–2) and $H(s)$ is as in eq. (P11.37–3), then the closed-loop system will be unstable if K is chosen *too* large. *Hint:* See Problem 11.34.

(c) Use root-locus techniques to show that if

$$G(s) = K(s + 100),$$

then the feedback system will be stable for all values of K sufficiently large if $H(s)$ is given by eq. (P11.37–1) *or* eq. (P11.37–3).

11.38. Consider the feedback system of Figure 11.3(b) with

$$H(z) = \frac{Kz^{-1}}{1 - z^{-1}}$$

and

$$G(z) = 1 - az^{-1}.$$

(a) Sketch the root locus for $K > 0$ and $K < 0$ when $a = 1/2$.

(b) Repeat part (a) when $a = -1/2$.

(c) With $a = -1/2$, find a value of K for which the closed-loop impulse response is of the form

$$(A + Bn)\alpha^n$$

for some values of the constants A, B, and α, with $|\alpha| < 1$. (*Hint:* What must the denominator of the closed-loop system function look like in this case?)

11.39. Consider the feedback system of Figure P11.39 with

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad G(z) = K. \qquad \text{(P11.39–1)}$$

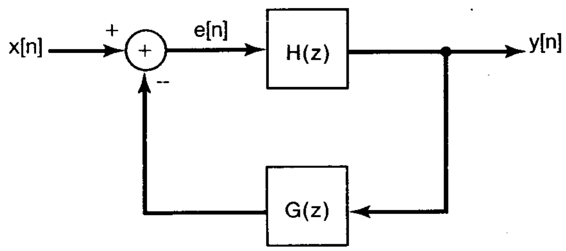


Figure P11.39

(a) Plot the root locus for $K > 0$.

(b) Plot the root locus for $K < 0$. (*Note:* Be careful with this root locus. By applying the angle criterion on the real axis, you will find that as K is decreased from zero, the closed loop approaches $z = +\infty$ along the positive real axis and then returns along the negative real axis from $z = -\infty$. Check that this is in fact the case by explicitly solving for the closed-loop pole as a function of K. At what value of K is the pole at $|z| = \infty$?)

(c) Find the full range of values of K for which the closed-loop system is stable.

(d) The phenomenon observed in part (b) is a direct consequence of the fact that in this example the numerator and denominator of $G(z)H(z)$ have the same degree. When this occurs in a discrete-time feedback system, it means that there is a delay-free loop in the system. That is, the output at a given point in time is being fed back into the system and in turn affects its own value at the same point in time. To see that this is the case in the system we are considering here, write the difference equation relating $y[n]$ and $e[n]$. Then write $e[n]$ in terms of the input and output for the feedback system. Contrast this result with that of the feedback system with

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad G(z) = Kz^{-1}. \qquad \text{(P11.39–2)}$$

The primary consequence of having delay-free loops is that such feedback systems cannot be implemented in the form depicted. For example, for the system of eq. (P11.39–1), we cannot first calculate $e[n]$ and then $y[n]$, because $e[n]$ depends on $y[n]$. Note that we *can* perform this type of calculation for the system of eq. (P11.39–2), since $e[n]$ depends on $y[n-1]$.

(e) Show that the feedback system of eq. (P11.39–1) represents a causal system, except for the value of K for which the closed-loop pole is at $|z| = \infty$.

11.40. Consider the discrete-time feedback system depicted in Figure P11.40. The system in the forward path is not very well damped, and we would like to choose the feedback system function so as to improve the overall damping. By using the root-locus method, show that this can be done with

$$G(z) = 1 - \frac{1}{2}z^{-1}.$$

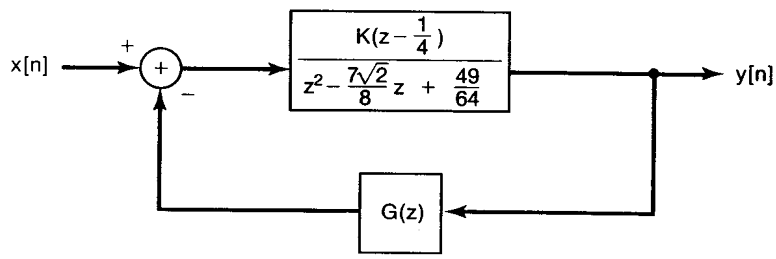


Figure P11.40

Specifically, sketch the root locus for $K > 0$, and specify the value of the gain K for which a significant improvement in damping is obtained.

11.41. **(a)** Consider a feedback system with

$$H(z) = \frac{z+1}{z^2+z+\frac{1}{4}}, \quad G(z) = \frac{K}{z-1}.$$

(i) Write the closed-loop system function explicitly as a ratio of two polynomials. (The denominator polynomial will have coefficients that depend on K.)

(ii) Show that the sum of the closed-loop poles is independent of K.

(b) More generally, consider a feedback system with system function

$$G(z)H(z) = K\frac{z^m + b_{m-1}z^{m-1} + \cdots + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_0}.$$

Show that if $m \leq n-2$, the sum of the closed-loop poles is independent of K.

11.42. Consider again the discrete-time feedback system of Example 11.3:

$$G(z)H(z) = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}.$$

The root loci for $K > 0$ and $K < 0$ are depicted in Figure 11.16.

(a) Consider the root locus for $K > 0$. In this case, the system becomes unstable when one of the closed-loop poles is less than or equal to -1. Find the value of K for which $z = -1$ is a closed-loop pole.

(b) Consider the root locus for $K < 0$. In this case, the system becomes unstable when one of the closed-loop poles is greater than or equal to 1. Find the value of K for which $z = 1$ is a closed-loop pole.

(c) What is the full range of values of K for which the closed-loop system is stable?

11.43. Consider a discrete-time feedback system with

$$G(z)H(z) = \frac{1}{z(z-1)}.$$

(a) Sketch the root locus for $K > 0$ and for $K < 0$.

(b) If you have sketched the root locus correctly for $K > 0$, you will see that the two branches of the root locus cross and exit from the unit circle. Consequently, we can conclude that the closed-loop system is stable for $0 < K < K_0$, where K_0 is the value of the gain for which the two branches intersect the unit circle. At what points on the unit circle do the branches exit from it? What is the value of K_0?

11.44. As mentioned in Section 11.4, the continuous-time Nyquist criterion can be extended to allow for poles of $G(s)H(s)$ on the $j\omega$-axis. In this problem, we will illustrate the general technique for doing this by means of several examples. Consider a continuous-time feedback system with

$$G(s)H(s) = \frac{1}{s(s+1)}. \qquad \text{(P11.44–1)}$$

When $G(s)H(s)$ has a pole at $s = 0$, we modify the contour of Figure 11.19 by avoiding the origin. To do this, we indent the contour by adding a semicircle of infinitesimal radius ϵ into the right-half plane. [See Figure P11.44(a).] Thus, only a small part of the right-half plane is not enclosed by the modified contour, and its area goes to zero as we let $\epsilon \to 0$. Consequently, as $M \to \infty$, the contour will enclose the entire right-half plane. As in the text, $G(s)H(s)$ is a constant (in this case zero) along the circle of infinite radius. Thus, to plot $G(s)H(s)$ along the contour, we need only plot it for the portion of the contour consisting of the $j\omega$-axis and the infinitesimal circle.

(a) Show that

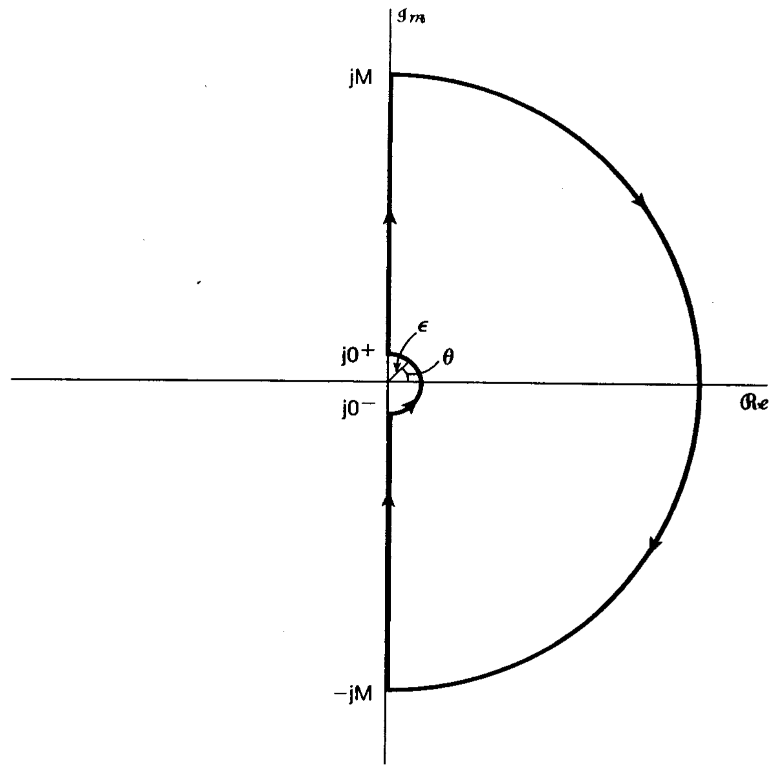
$$\measuredangle G(j0^+)H(j0^+) = -\frac{\pi}{2}$$

and

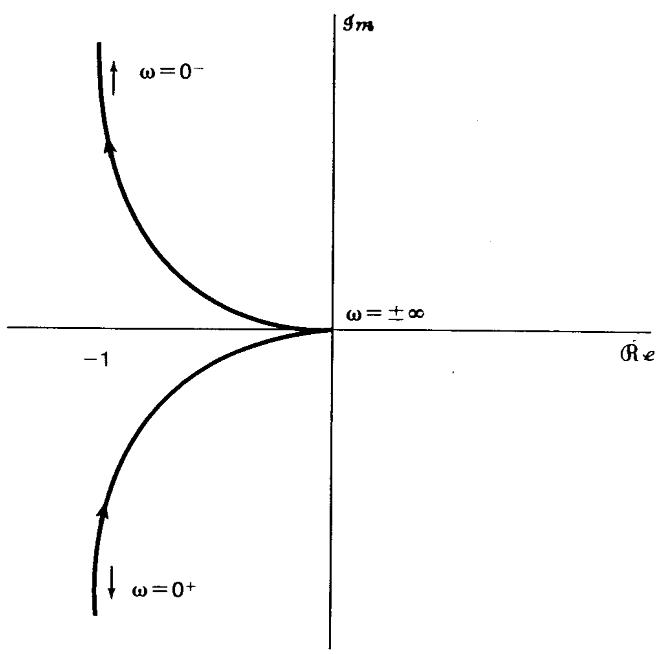
$$\measuredangle G(j0^-)H(j0^-) = \frac{\pi}{2},$$

where $s = j0^-$ is the point where the infinitesimal semicircle meets the $j\omega$-axis just below the origin and $s = j0^+$ is the corresponding point just above the origin.

(b) Use the result of part (a) together with eq. (P11.44–1) to verify that Figure P11.44(b) is an accurate sketch of $G(s)H(s)$ along the portions of the contour



(a)



(b)

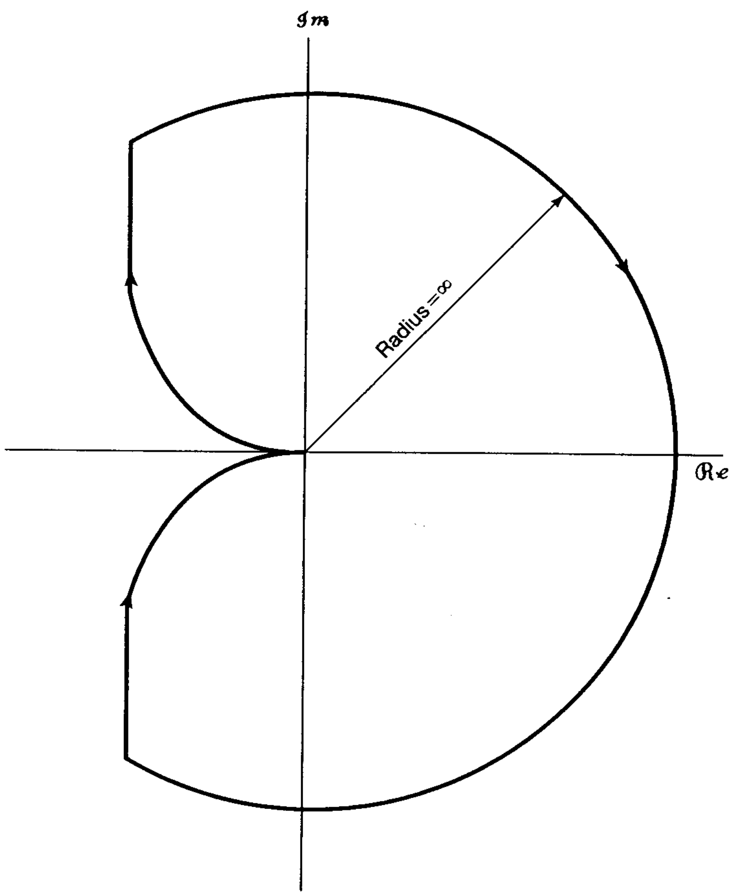
Figure P11.44

from $-j\infty$ to $j0^-$ and $j0^+$ to $j\infty$. In particular, check that $\sphericalangle G(j\omega)H(j\omega)$ and $|G(j\omega)H(j\omega)|$ behave in the manner depicted in the figure.

(c) All that remains to be done is to determine the plot of $G(s)H(s)$ along the small semicircle about $s = 0$. Note that as $\epsilon \to 0$, the magnitude of $G(s)H(s)$ along this contour goes to infinity. Show that as $\epsilon \to 0$, the contribution of the pole at $s = -1$ to $\sphericalangle G(s)H(s)$ along the semicircle is zero. Then show that as $\epsilon \to 0$,

$$\sphericalangle G(s)H(s) = -\theta,$$

where θ is as defined in Figure P11.44(a). Thus, since θ varies from $-\pi/2$ at $s = j0^-$ to $+\pi/2$ at $s = j0^-$ in the counterclockwise direction, $\sphericalangle G(s)H(s)$ must go from $+\pi/2$ at $s = j0^+$ to $-\pi/2$ at $s = j0^+$ in the clockwise direction. The result is the complete Nyquist plot depicted in Figure P11.44(c).



(c)

Figure P11.44 *Continued*

(d) Using the Nyquist plot of Figure P11.44(c), find the range of values of K for which the closed-loop feedback system is stable. (*Note:* As presented in the text, the continuous-time Nyquist criterion states that, for closed-loop system stability, the net number of clockwise encirclements of the point $-1/K$ must equal minus the net number of right-half plane poles of $G(s)H(s)$. In the present example, note that the pole of $G(s)H(s)$ at $s = 0$ is ***outside*** the modified contour. Consequently, it is *not* included in counting the poles of $G(s)H(s)$ in the right-half plane [i.e., only poles of $G(s)H(s)$ strictly ***inside*** the right-half plane are counted in applying the Nyquist criterion]. Thus, in this case, since $G(s)H(s)$ has no poles strictly inside the right-half plane, we must have ***no*** encirclements of the point $s = -1/K$ for closed-loop system stability.)

(e) Follow the steps outlined in parts (a)–(c) to sketch the Nyquist plots for each of the following:

(i) $G(s)H(s) = \frac{(s/10)+1}{s(s+1)}$

(ii) $G(s)H(s) = \frac{1}{s(s+1)^2}$

(iii) $G(s)H(s) = \frac{1}{s^2}$ [be careful in calculating $\sphericalangle G(s)H(s)$ along the infinitesimal semicircle]

(iv) $G(s)H(s) = \frac{s+1}{s(1-s)}$ [be careful in calculating $\sphericalangle G(j\omega)H(j\omega)$ as ω is varied; make sure to take the minus sign in the denominator into account]

(v) $G(s)H(s) = \frac{s+1}{s^2}$ [same remark as for (iii)]

In each case, use the Nyquist criterion to determine the range of values of K (if any such range exists) for which the closed-loop system is stable. Also, use another method (root locus or direct calculation of the closed-loop poles as a function of K) to provide a partial check of the correctness of your Nyquist plot. [*Note:* In sketching the Nyquist plots, you may find it useful to sketch the Bode plots of $G(s)H(s)$ first. It may also be helpful to determine the values of ω for which $G(j\omega)H(j\omega)$ is real.]

(f) Repeat part (e) for:

(i) $G(s)H(s) = \frac{1}{s^2+1}$

(ii) $G(s)H(s) = \frac{s+1}{s^2+1}$

Note: In these cases there are *two* poles on the imaginary axis; accordingly, you will need to modify the contour of Figure 11.19 to avoid each of them. Use infinitesimal semicircles, as in Figure P11.44(a).

11.45. Consider a system with system function

$$H(s) = \frac{1}{(s+1)(s-2)}. \qquad \text{(P11.45–1)}$$

Because this system is unstable, we would like to devise some method for its stabilization.

(a) Consider first a series compensation scheme as illustrated in Figure P11.45(a). Show that the overall system of this figure is stable if the system function

$$C(s) = \frac{s-2}{s+3}.$$

In practice, this is *not* considered to be a particularly useful way to attempt to stabilize a system. Explain why.

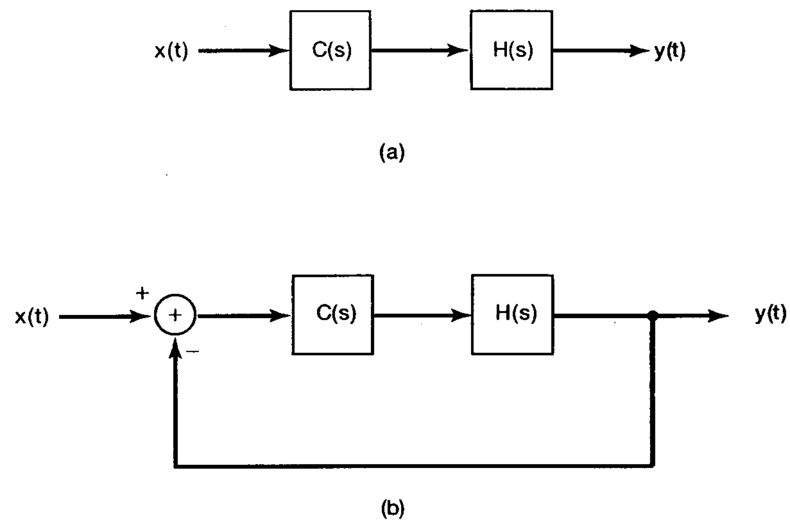


Figure P11.45

(b) Suppose that instead we use a feedback system, as depicted in Figure P11.45(b). Is it possible to stabilize this system using a constant gain, that is,

$$C(s) = K,$$

for the stabilizing element? Justify your answer using Nyquist techniques.

(c) Show that the system of Figure P11.45(b) can be stabilized if $C(s)$ is a proportional plus derivative system—that is, if

$$C(s) = K(s + a).$$

Consider both the case $0 < a < 1$ and the case $a > 1$.

(d) Suppose that

$$C(s) = K(s + 2).$$

Choose the value of K such that the closed-loop system has a pair of complex poles with a damping ratio $\zeta = 1/2$. (*Hint:* In this case, the denominator of the closed-loop system must have the form

$$s^2 + \omega_n s + \omega_n^2$$

for some value of $\omega_n > 0$.)

(e) Pure derivative compensation is both impossible to obtain and undesirable in practice. This is because the required amplification of arbitrarily high frequencies neither can be obtained nor is advisable, as all real systems are subject to some level of high-frequency disturbances. Thus, suppose that we consider a compensator of the form

$$C(s) = K\left(\frac{s + a}{s + b}\right), \quad a, b > 0. \qquad \text{(P11.45–2)}$$

If $b < a$, this is a *lag network*: $\measuredangle C(j\omega) < 0$ for all $\omega > 0$, so that the phase of the output of the system lags the phase of the input. If $b > a$, $\measuredangle C(j\omega) > 0$ for all $\omega > 0$, and the system is then called a *lead network*.

(i) Show that it is possible to stabilize the system with the lead compensator

$$C(s) = K\frac{s + \frac{1}{2}}{s + 2} \quad \text{(P11.45–3)}$$

if K is chosen large enough.

(ii) Show that it is not possible to stabilize the feedback system of Figure P11.45(b) using the lag network

$$C(s) = K\frac{s + 3}{s + 2}.$$

Hint: Use the results of Problem 11.34 in sketching the root locus. Then determine the points on the $j\omega$-axis that are on the root locus and the values of K for which each of these points is a closed-loop pole. Use this information to prove that for no value of K are *all* of the closed-loop poles in the left-half plane.

11.46. Consider the continuous-time feedback system depicted in Figure P11.46(a).

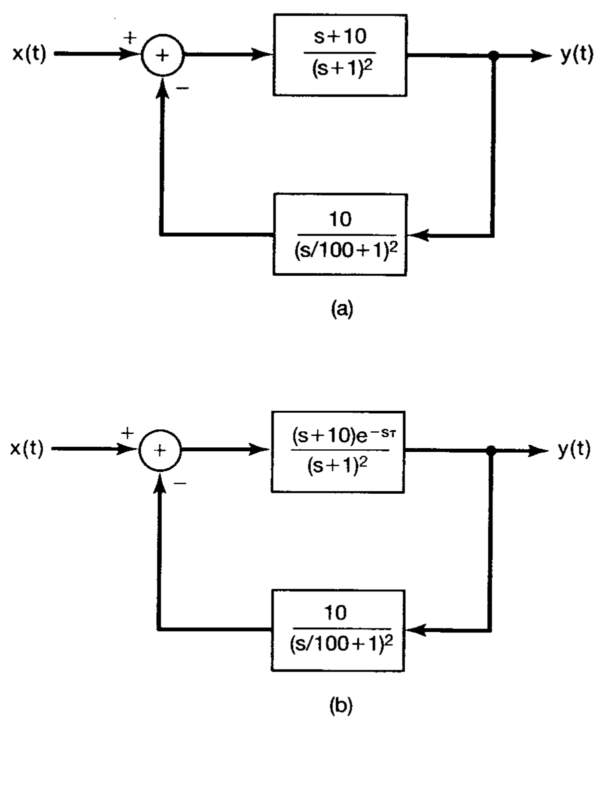


Figure P11.46

(a) Use the straight-line approximations to Bode plots developed in Chapter 6 to obtain a sketch of the log magnitude-phase plot of this system. Estimate the phase and gain margins from your plot.

(b) Suppose that there is an unknown delay within the feedback system, so that the actual feedback system is as shown in Figure P11.46(b). Approximately what is the largest delay τ that can be tolerated before the feedback system becomes unstable? Use your results from part (a) for this calculation.

(c) Calculate more precise values of the phase and gain margins, and compare these to your results in part (a). This should give you some idea of the size of the errors that are incurred in using the approximate Bode plots.

11.47. As mentioned at the end of Section 11.5, the phase and gain margins may provide *sufficient* conditions to ensure that a stable feedback system remains stable. For example, we showed that a stable feedback system will remain stable as the gain is increased, until we reach a limit specified by the gain margin. This does *not* imply (a) that the feedback system cannot be made unstable by *decreasing* the gain or (b) that the system will be unstable for *all* values of gain greater than the gain margin limit. In this problem, we illustrate these two points.

(a) Consider a continuous-time feedback system with

$$G(s)H(s) = \frac{1}{(s-1)(s+2)(s+3)}.$$

Sketch the root locus for this system for $K > 0$. Use the properties of the root locus described in the text and in Problem 11.34 to help you draw the locus accurately. Once you do so, you should see that for small values of the gain K the system is unstable, for larger values of K the system is stable, while for still larger values of K the system again becomes unstable. Find the range of values of K for which the system is stable. *Hint:* Use the same method as is employed in Example 11.2 and Problem 11.35 to determine the values of K at which branches of the root locus pass through the origin and cross the $j\omega$-axis.

If we set our gain somewhere within the stable range that you have just found, we can increase the gain somewhat and maintain stability, but a large enough increase in gain causes the system to become unstable. This maximum amount of increase in gain at which the closed-loop system just becomes unstable is the gain margin. Note that if we *decrease* the gain too much, we can also cause instability.

(b) Consider the feedback system of part (a) with the gain K set at a value of 7. Show that the closed-loop system is stable. Sketch the log magnitude-phase plot of this system, and show that there are two nonnegative values of ω for which $\sphericalangle G(j\omega)H(j\omega) = -\pi$. Further, show that, for one of these values $7|G(j\omega)H(j\omega)| < 1$, and for the other $7|G(j\omega)H(j\omega)| > 1$. The first value provides us with the usual gain margin—that is, the factor $1/|7G(j\omega)H(j\omega)|$ by which we can increase the gain and cause instability. The second provides us with the factor $1/|7G(j\omega)H(j\omega)|$ by which we can decrease the gain and just cause instability.

(c) Consider a feedback system with

$$G(s)H(s) = \frac{(s/100 + 1)^2}{(s+1)^3}.$$

Sketch the root locus for $K > 0$. Show that two branches of the root locus begin in the left-half plane and, as K is increased, move into the right-half plane and then back into the left-half plane. Do this by examining the equation

$$G(j\omega)H(j\omega) = -\frac{1}{K}.$$

Specifically, by equating the real and imaginary parts of this equation, show that there are two values of $K \geq 0$ for which the closed-loop poles lie on the $j\omega$-axis.

Thus, if we set the gain at a small enough value so that the system is stable, then we can increase the gain up until the point at which the two branches of the root locus intersect the $j\omega$-axis. For a range of values of gain beyond this point, the closed-loop system is unstable. However, if we *continue* to increase the gain, the system will again become stable for K large enough.

(d) Sketch the Nyquist plot for the system of part (c), and confirm the conclusions reached in part (c) by applying the Nyquist criterion. (Make sure to count the *net* number of encirclements of $-1/K$.)

Systems such as that considered in parts (c) and (d) of this problem are often referred to as being *conditionally stable* systems, because their stability properties may change several times as the gain is varied.

11.48. In this problem, we illustrate the discrete-time counterpart of the technique described in Problem 11.44. Specifically, the discrete-time Nyquist criterion can be extended to allow for poles of $G(z)H(z)$ on the unit circle.

Consider a discrete-time feedback system with

$$G(z)H(z) = \frac{z^{-2}}{1 - z^{-1}} = \frac{1}{z(z-1)}. \quad \text{(P11.48–1)}$$

In this case, we modify the contour on which we evaluate $G(z)H(z)$, as illustrated in Figure P11.48(a).

(a) Show that

$$\sphericalangle G(e^{j0^+})H(e^{j0^+}) = -\frac{\pi}{2}$$

and

$$\sphericalangle G(e^{j2\pi^-})H(e^{j2\pi^-}) = \frac{\pi}{2},$$

where $z = e^{j2\pi^-}$ is the point below the real axis at which the small semicircle intersects the unit circle and $z = e^{j0^+}$ is the corresponding point above the real axis.

(b) Use the results of part (a) together with eq. (P11.48–1) to verify that Figure P11.48(b) is an accurate sketch of $G(z)H(z)$ along the portion of the

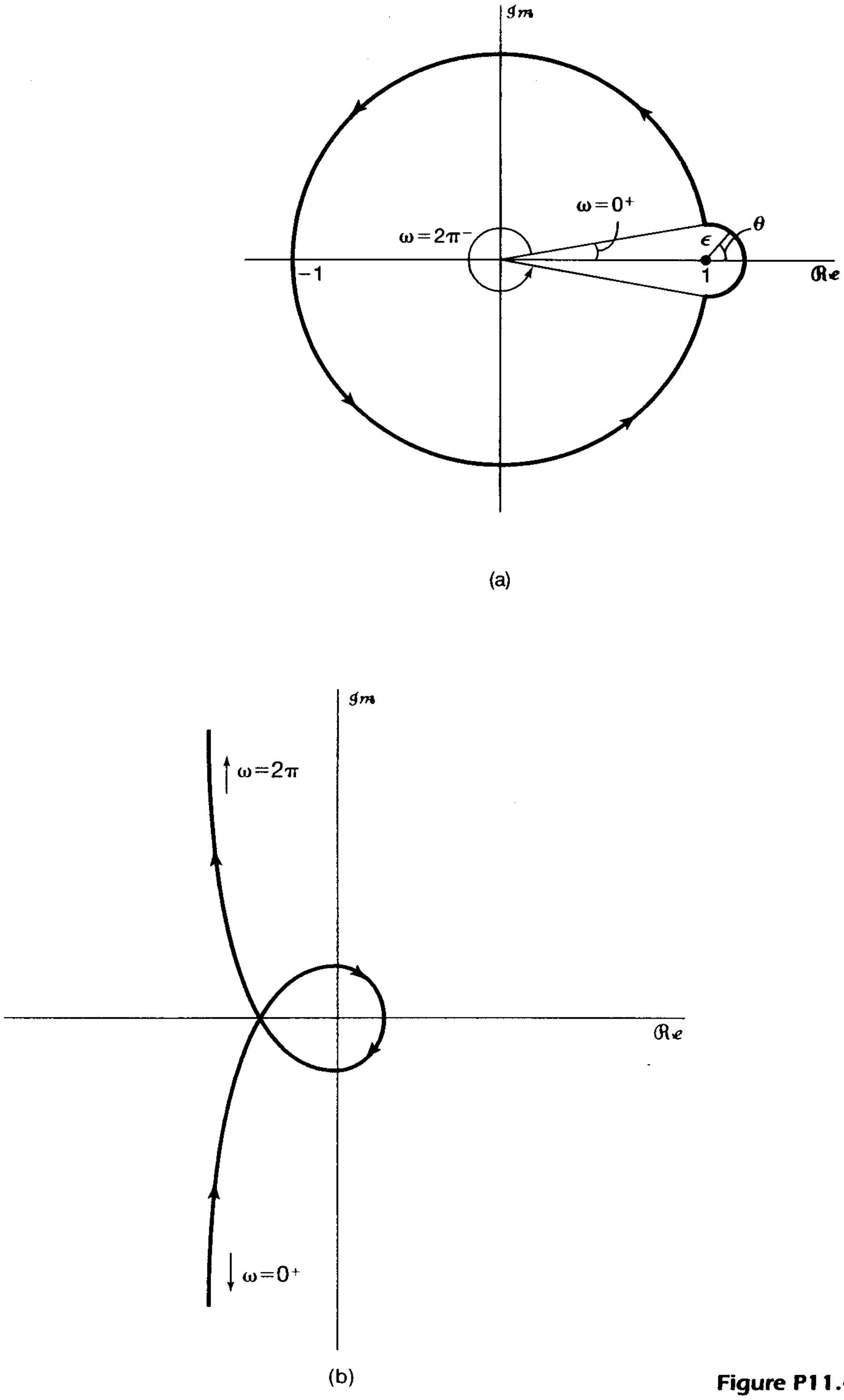


Figure P11.48

contour $z = e^{j\omega}$ as ω varies from 0^+ to $2\pi^-$ in a counterclockwise direction. In particular, verify that the angular variation of $G(e^{j\omega})H(e^{j\omega})$ is as indicated.

(c) Find the value of ω for which $\sphericalangle G(e^{j\omega})H(e^{j\omega}) = -\pi$, and verify that

$$|G(e^{j\omega})H(e^{j\omega})| = 1$$

at this point. [*Hint:* Use the geometrical method for evaluating $\sphericalangle G(e^{j\omega})H(e^{j\omega})$ together with some elementary geometry to determine the value of ω.]

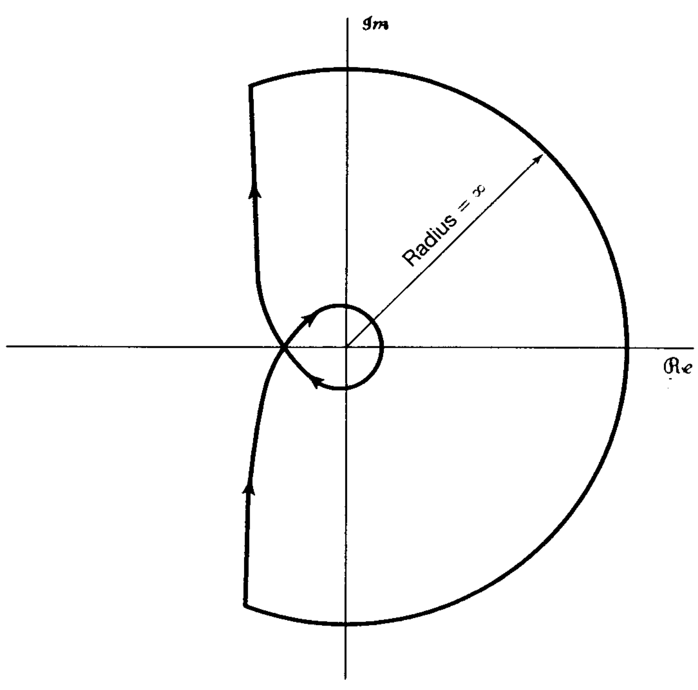
(d) Consider next the plot of $G(z)H(z)$ along the small semicircle about $z = 1$. Note that as $\epsilon \to 0$, the magnitude of $G(z)H(z)$ along this contour goes to infinity. Show that as $\epsilon \to 0$, the contribution of the pole at $z = 0$ to $\sphericalangle G(z)H(z)$ along the semicircle is zero. Then show that as $\epsilon \to 0$,

$$\sphericalangle G(z)H(z) = -\theta,$$

where θ is as defined in Figure P11.48(a).

Thus, since θ varies from $-\pi/2$ to $+\pi/2$ in the counterclockwise direction, $\sphericalangle G(z)H(z)$ varies from $+\pi/2$ to $-\pi/2$ in the clockwise direction. The result is the complete Nyquist plot of Figure P11.48(c).

(e) Using the Nyquist plot, find the range of values of K for which the closed-loop feedback system is stable. [*Note:* Since the pole of $G(z)H(z)$ at $z = 1$ is



(c)

Figure P11.48 *Continued*

inside the modified contour, it is not included in counting the poles of $G(z)H(z)$ outside the unit circle. That is, only poles *strictly outside* the unit circle are counted in applying the Nyquist criterion. Thus, in this case, since $G(z)H(z)$ has no poles strictly outside the unit circle, we must have no encirclements of the point $z = -1/K$ for closed-loop stability.]

(f) Follow the steps outlined in parts (a), (b), and (d) to sketch the Nyquist plots for each of the following:

(i) $\frac{z+\frac{1}{2}+\sqrt{3}}{z-1}$

(ii) $\frac{1}{(z-1)(z+\frac{1}{2}+\sqrt{3})}$

(iii) $\frac{z+1}{z(z-1)}$

(iv) $\frac{z-1/\sqrt{3}}{(z-1)^2}$ [be careful in calculating $\sphericalangle G(z)H(z)$ along the infinitesimal semicircle]

For each of the preceding, use the Nyquist criterion to determine the range of values of K (if any such range exists) for which the closed-loop system is stable. Also, use another method (root locus or direct calculation of the closed-loop poles as a function of K) to provide a partial check of the correctness of your Nyquist plot. *Note:* In sketching the Nyquist plots, you may find it useful to first sketch the magnitude and phase plots as a function of frequency or at least calculate $|G(e^{j\omega})H(e^{j\omega})|$ and $\sphericalangle G(e^{j\omega})H(e^{j\omega})$ at several points. Also, it is helpful to determine the values of ω for which $G(e^{j\omega})H(e^{j\omega})$ is real.

(g) Repeat part (f) for

$$G(z)H(z) = \frac{1}{z^2 - 1}.$$

In this case there are two poles on the unit circle, and thus, you must modify the contour around each of these by including an infinitesimal semicircle that extends outside the unit circle, thereby placing the pole inside the contour.

EXTENSION PROBLEMS

11.49. In this problem, we provide an illustration of how feedback can be used to increase the bandwidth of an amplifier. Consider an amplifier whose gain falls off at high frequencies. That is, suppose the system function of this amplifier is

$$H(s) = \frac{Ga}{s+a}.$$

(a) What is the dc gain of the amplifier (i.e., the magnitude of its frequency response at 0 frequency)?

(b) What is the system time constant?

(c) Suppose we define the bandwidth of the system as the frequency at which the magnitude of the amplifier frequency response is $1/\sqrt{2}$ times its magnitude at dc. What is the bandwidth of the amplifier?

(d) Suppose we place the amplifier in a feedback loop as depicted in Figure P11.49. What is the dc gain of the closed-loop system? What are the time constant and the bandwidth of the closed-loop system?

(e) Find the value of K that leads to a closed-loop bandwidth that is exactly double the bandwidth of the open-loop amplifier. What are the corresponding closed-loop system time constant and dc gain?

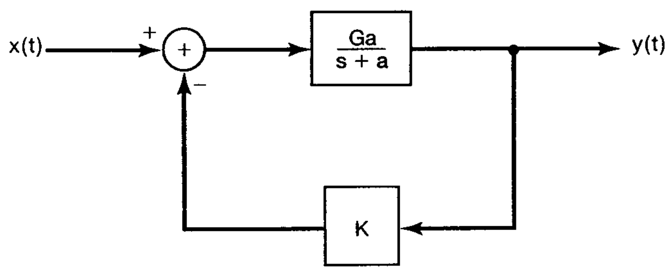


Figure P11.49

11.50. As mentioned in the text, an important class of devices used in the implementation of feedback systems is the class of operational amplifiers. A model for such an amplifier is depicted in Figure P11.50(a). The amplifier's input is the difference

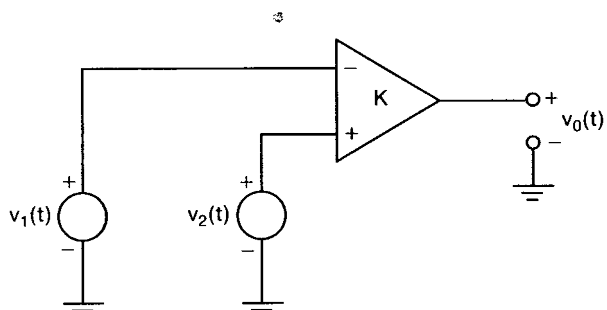


Figure P11.50a

between two voltages $v_2(t)$ and $v_1(t)$, and the output voltage is an amplified version of the input; that is,

$$v_o(t) = K[v_2(t) - v_1(t)]. \qquad \text{(P11.50–1)}$$

Consider an operational amplifier connection shown in Figure P11.50(b). In this figure, $Z_1(s)$ and $Z_2(s)$ are impedances. (That is, each is the system function of an LTI system whose input is the current flowing through the impedance element and whose output is the voltage across the element.) Making the approximation that the input impedance of the operational amplifier is infinite and that its output impedance is zero, we obtain the following relationship between $V_1(s)$, $V_i(s)$, and $V_o(s)$, the Laplace transforms of $v_1(t)$, $v_i(t)$, and $v_o(t)$, respectively:

$$V_1 = \left[\frac{Z_2(s)}{Z_1(s) + Z_2(s)}\right] V_i(s) + \left[\frac{Z_1(s)}{Z_1(s) + Z_2(s)}\right] V_o(s). \qquad \text{(P11.50–2)}$$

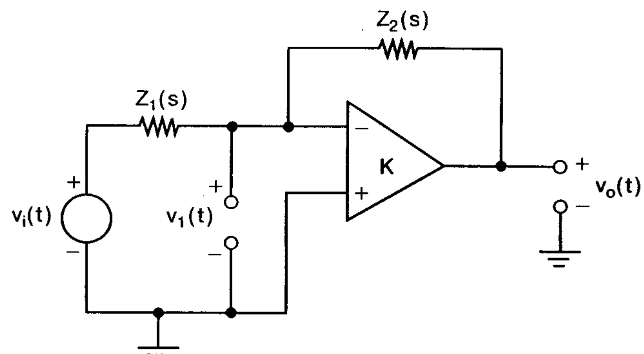


Figure P11.50b

Also, from eq. (P11.50–1) and Figure P11.50(b), we see that

$$V_o(s) = -KV_1(s). \qquad \text{(P11.50–3)}$$

(a) Show that the system function

$$H(s) = \frac{V_o(s)}{V_i(s)}$$

for the interconnection of Figure P11.50(b) is identical to the overall closed-loop system function for the system of Figure P11.50(c).

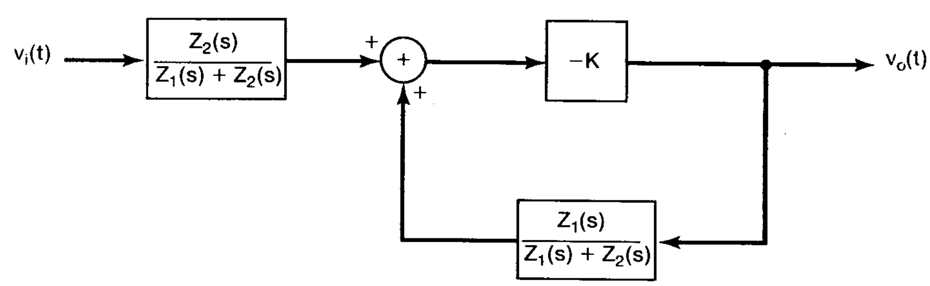


Figure P11.50c

(b) Show that if $K >> 1$, then

$$H(s) \approx -\frac{Z_2(s)}{Z_1(s)}.$$

11.51. **(a)** Suppose that in Figure P11.50(b) $Z_1(s)$ and $Z_2(s)$ are both pure resistances, say, R_1 and R_2, respectively. A typical value for R_2/R_1 is in the range 1 to 10^3, while a typical value for K is 10^6. Using the results of Problem 11.50(a), calculate the actual system function for this value of K and for R_2/R_1 equal to 1 and then to 10^3, and compare each resulting value to $-R_2/R_1$. This should give you some idea of how good the approximation of Problem 11.50(b) typically is.

(b) One of the important uses of feedback is in the reduction of system sensitivity to variations in parameters. This is particularly important for circuits involving operational amplifiers, which have high gains that may be known only approximately.

(i) Consider the circuit discussed in part (a), with $R_2/R_1 = 10^2$. What is the percentage change in the closed-loop gain of the system if K changes from 10^6 to 5×10^5?

(ii) How large must K be so that a 50% reduction in its value results in only a 1% reduction in the closed-loop gain? Again, take $R_2/R_1 = 10^2$.

11.52. Consider the circuit of Figure P11.52. This circuit is obtained by using

$$Z_1(s) = R, \quad Z_2(s) = \frac{1}{Cs}$$

in Figure P11.50(b). Using the results from Problem 11.50, show that the system behaves approximately like an integrator. In what frequency range (expressed in terms of K, R, and C) does this approximation break down?

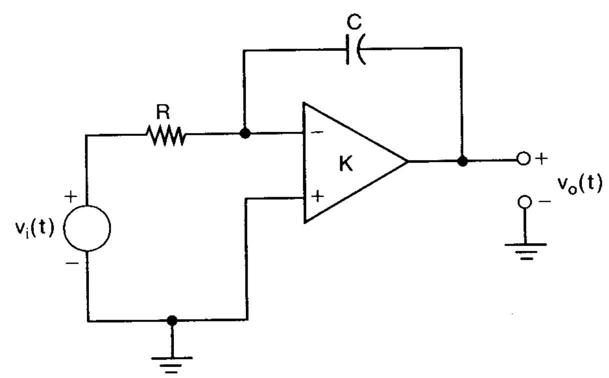


Figure P11.52

11.53. Consider the circuit depicted in Figure P11.53(a), which is obtained from the circuit of Figure P11.50(b) by using $Z_1(s) = R$ and by replacing $Z_2(s)$ with a diode that has an exponential current-voltage relationship. Assume that this relationship is of the form

$$i_d(t) = Me^{qv_d(t)/kT}, \qquad \text{(P11.53–1)}$$

where M is a constant that depends upon the construction of the diode, q is the charge of an electron, k is Boltzmann's constant, and T is absolute temperature. Note that the idealized relationship of eq. (P11.53–1) assumes that there is no possibility of a negative diode current. Usually, there *is* some small maximum negative value of diode current, but we will neglect this possibility in our analysis.

(a) Assuming that the input impedance of the operational amplifier is infinite and that its output impedance is zero, show that the following relations hold:

$$v_o(t) = v_d(t) + Ri_d(t) + v_i(t), \qquad \text{(P11.53–2)}$$

$$v_o(t) = -K[v_o(t) - v_d(t)]. \qquad \text{(P11.53–3)}$$

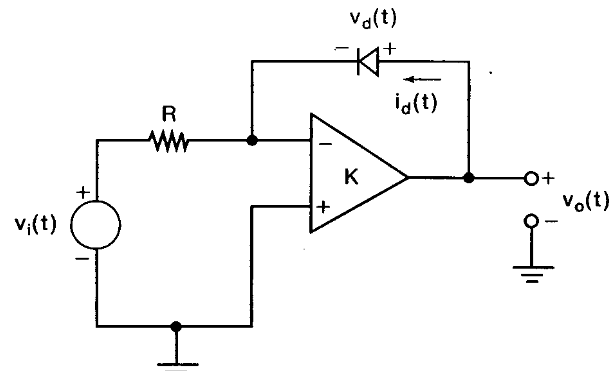


Figure P11.53a

(b) Show that for K large, the relationship between $v_o(t)$ and $v_i(t)$ is essentially the same as in the feedback system of Figure P11.53(b), in which the system in the feedback path is a nonlinear memoryless system with input $v_o(t)$ and output

$$w(t) = RMe^{qv_o(t)/kT}.$$

(c) Show that for K large,

$$v_o(t) \approx \frac{kT}{q}\ln\left(-\frac{v_i(t)}{RM}\right). \qquad \text{(P11.53–4)}$$

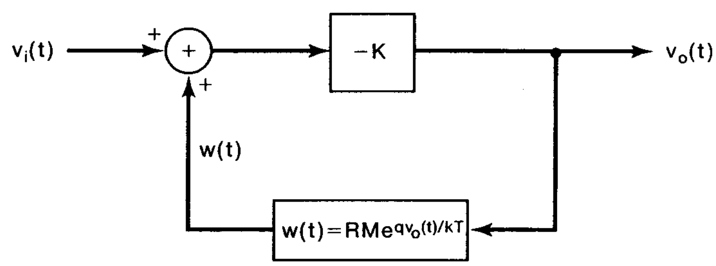


Figure P11.53b

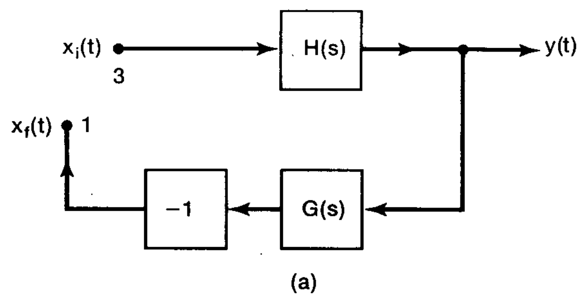
Note that eq. (P11.53–4) makes sense only for a negative $v_i(t)$, which is consistent with the requirement that the diode current cannot be negative. If a positive $v_i(t)$ is applied, the current $i_d(t)$ cannot balance the current through the resistor. Thus, a nonnegligible current is fed into the amplifier, causing it to saturate.

11.54. In this problem, we explore the use of positive feedback for generating oscillating signals.

(a) Consider the system illustrated in Figure P11.54(a). Show that $x_f(t) = x_i(t)$ if

$$G(s)H(s) = -1. \qquad \text{(P11.54–1)}$$

Suppose that we connect terminals 1 and 2 in Figure P11.54(a) and make $x_i(t) = 0$. Then the output of the system should remain unchanged if we



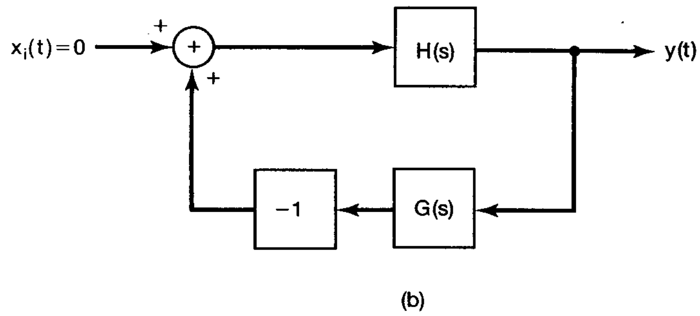


Figure P11.54

satisfy eq. (P11.54–1). The system now produces an output without any input. Therefore, the system shown in Figure P11.54(b) is an oscillator, provided that eq. (P11.54–1) is satisfied.

(b) A commonly used oscillator in practice is the sinusoidal oscillator. For such an oscillator, we may rewrite the condition of eq. (P11.54–1) as

$$G(j\omega_0)H(j\omega_0) = -1. \qquad \text{(P11.54–2)}$$

What is the value of the closed-loop gain for the system shown in Figure P11.54(b) at ω_0 when eq. (P11.54–2) is satisfied?

(c) A sinusoidal oscillator may be constructed on the basis of the principle outlined above by using the circuit shown in Figure P11.54(c). The input to the

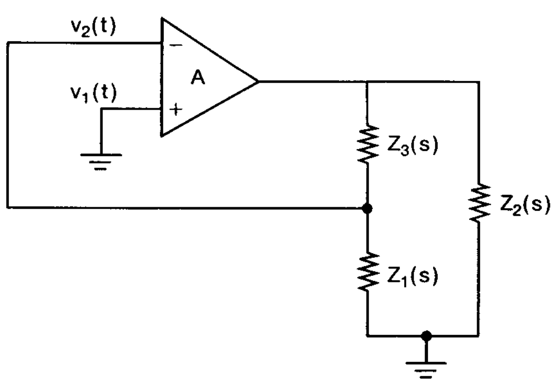


Figure P11.54c

amplifier is the difference between the voltages $v_1(t)$ and $v_2(t)$. In this circuit, the amplifier has a gain of A and an output resistance of R_0. $Z_1(s)$, $Z_2(s)$, and $Z_3(s)$ are impedances. (That is, each is the system function of an LTI system whose input is the current flowing through the impedance element and whose output is the voltage across the element.) It can be shown that, for this circuit,

$$H(s) = \frac{-AZ_L(s)}{Z_L(s) + R_0},$$

where

$$Z_L = \frac{Z_2(s)(Z_1(s) + Z_3(s))}{Z_1(s) + Z_2(s) + Z_3(s)}.$$

Also, we can show that

$$G(s) = \frac{-Z_1(s)}{Z_1(s) + Z_3(s)}.$$

(i) Show that

$$G(s)H(s) = \frac{AZ_1(s)Z_2(s)}{R_0(Z_1(s) + Z_2(s) + Z_3(s)) + Z_2(s)(Z_1(s) + Z_3(s))}.$$

(ii) If $Z_1(s)$, $Z_2(s)$, and $Z_3(s)$ are pure reactances (i.e., inductances or capacitances), we can write $Z_1(j\omega) = jX_1(j\omega)$, $Z_2(j\omega) = jX_2(j\omega)$, and $Z_3(j\omega) = jX_3(j\omega)$, where $X_i(j\omega)$, $i = 1, 2, 3$, are all real. Using the results of parts (b) and (i), show that a necessary condition for the circuit to produce oscillations is

$$X_1(j\omega) + X_2(j\omega) + X_3(j\omega) = 0.$$

(iii) Show also that, in addition to the constraint of part (ii), the constraint $AX_1(j\omega) = X_2(j\omega)$ has to be satisfied for the circuit to produce oscillations. [Since $X_i(j\omega)$ is positive for inductances and negative for capacitances, the latter constraint requires that $Z_1(s)$ and $Z_2(s)$ be reactances of the same type (i.e., both should be inductances or both should be capacitances).]

(iv) Let us assume that $Z_1(s)$ and $Z_2(s)$ are both inductances such that

$$X_1(j\omega) = X_2(j\omega) = \omega L.$$

Let us also assume that

$$X_3(j\omega) = -1/(\omega C)$$

is a capacitance. Use the condition derived in (ii) to determine the frequency (in terms of L and C) at which the circuit oscillates.

11.55. (a) Consider the nonrecursive discrete-time LTI filter depicted in Figure P11.55(a). Through the use of feedback around this nonrecursive system, a recursive filter can be implemented. To do so, consider the configuration shown in Figure P11.55(b), in which $H(z)$ is the system function of the nonrecursive LTI system of Figure P11.55(a). Determine the overall system function of this feedback system, and find the difference equation relating the input to the output of the overall system.

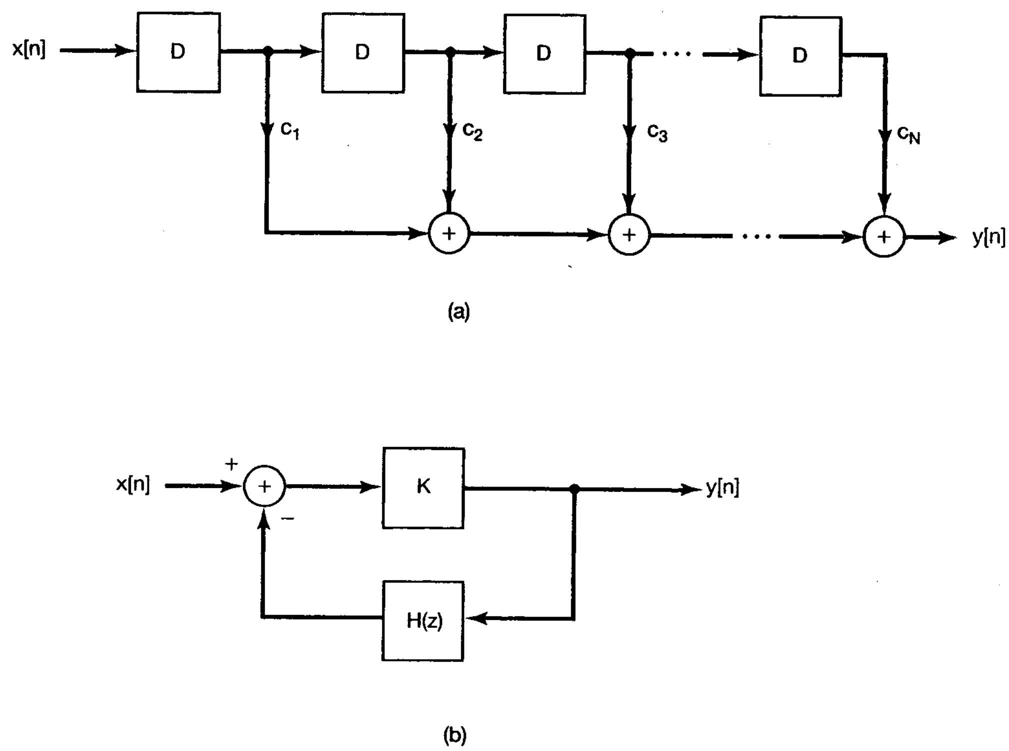


Figure P11.55

(b) Now suppose that $H(z)$ in Figure P11.55(b) is the system function of a recursive LTI system. Specifically, suppose that

$$H(z) = \frac{\sum_{i=1}^{N} c_i z^{-i}}{\sum_{i=1}^{N} d_i z^{-i}}.$$

Show how one can find values of the coefficients $K, c_1, \ldots, c_N$, and $d_0, \ldots, d_N$, such that the closed-loop system function is

$$Q(z) = \frac{\sum_{i=0}^{N} b_i z^{-i}}{\sum_{i=0}^{N} a_i z^{-i}},$$

where the a_i and b_i are specified coefficients.

In this problem, we have seen that the use of feedback provides us with alternative implementations of LTI systems specified by linear constant-coefficient difference equations. The implementation in part (a), consisting of feedback around a nonrecursive system, is particularly interesting, as some technologies are ideally suited to implementing tapped delay-line structures (i.e., systems consisting of chains of delays with taps at each delay whose outputs are weighted and then summed).

11.56. Consider an inverted pendulum mounted on a movable cart, as depicted in Figure P11.56. Here, we have modeled the pendulum as consisting of a massless rod of length L with a mass m attached at the end. The variable $\theta(t)$ denotes the pendulum's angular deflection from the vertical, g is gravitational acceleration, $s(t)$ is the position of the cart with respect to some reference point, $a(t)$ is the acceleration of the cart, and $x(t)$ represents the angular acceleration resulting from any disturbances, such as gusts of wind.

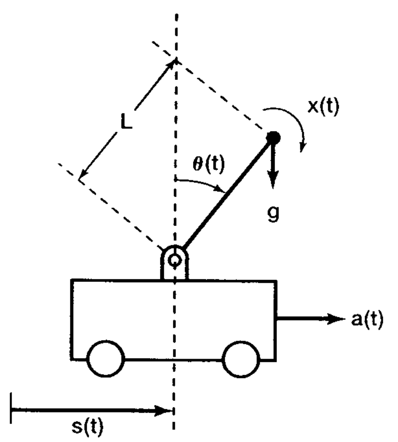


Figure P11.56

Our goal in this problem is to analyze the dynamics of the inverted pendulum and, more specifically, to investigate the problem of balancing the pendulum by a judicious choice of the acceleration $a(t)$ of the cart. The differential equation relating $\theta(t)$, $a(t)$, and $x(t)$ is

$$L\frac{d^2\theta(t)}{dt^2} = g\sin[\theta(t)] - a(t)\cos[\theta(t)] + Lx(t). \qquad \text{(P11.56–1)}$$

This relation merely equates the actual acceleration of the mass along a direction perpendicular to the rod to the applied accelerations [gravity, the disturbance acceleration due to $x(t)$, and the cart's acceleration] along this direction.

Note that eq. (P11.56–1) is a nonlinear differential equation. The detailed, exact analysis of the behavior of the pendulum requires that we examine this equation; however, we can obtain a great deal of insight into the dynamics of the pendulum by performing a linearized analysis. Specifically, let us examine the dynamics of the pendulum when it is nearly vertical [i.e., when $\theta(t)$ is small]. In this case, we can make the approximations

$$\sin[\theta(t)] \approx \theta(t), \cos[\theta(t)] \approx 1. \tag{P11.56–2}$$

(a) Suppose that the cart is stationary [i.e., $a(t) = 0$], and consider the causal LTI system with input $x(t)$ and output $\theta(t)$ described by eq. (P11.56–1), together with the approximations given in eq. (P11.56–2). Find the system function for this system, and show that it has a pole in the right-half of the plane, implying that the system is unstable.

(b) The result of part (a) indicates that if the cart is stationary, any minor angular disturbance caused by $x(t)$ will lead to growing angular deviations from the vertical. Clearly, at some point, these deviations will become sufficiently large so that the approximations of eq. (P11.56–2) will no longer be valid. At this point the linearized analysis is no longer accurate, but the fact that it is accurate for small angular displacements allows us to conclude that the vertical equilibrium position is unstable, since small angular displacements will grow rather than diminish.

We now wish to consider the problem of stabilizing the vertical position of the pendulum by moving the cart in an appropriate fashion. Suppose we try proportional feedback—that is,

$$a(t) = K\theta(t).$$

Assume that $\theta(t)$ is small, so that the approximations in eq. (P11.56–2) are valid. Draw a block diagram of the linearized system with $\theta(t)$ as the output, $x(t)$ as the external input, and $a(t)$ as the signal that is fed back. Show that the resulting closed-loop system is unstable. Find a value of K such that if $x(t) = \delta(t)$, the pendulum will sway back and forth in an undamped oscillatory fashion.

(c) Consider using the proportional-plus-derivative (PD) feedback,

$$a(t) = K_1\theta(t) + K_2\frac{d\theta(t)}{dt}.$$

Show that one can find values of K_1 and K_2 that stabilize the pendulum. In fact, using

$$g = 9.8 \text{ m/sec}^2$$

and (P11.56–3)

$$L = 0.5 \text{ m},$$

choose values of K_1 and K_2 so that the damping ratio of the closed loop system is 1 and the natural frequency is 3 rad/sec.

11.57. In this problem, we consider several examples of the design of tracking systems. Consider the system depicted in Figure P11.57. Here, $H_p(s)$ is the system whose output is to be controlled, and $H_c(s)$ is the compensator to be designed. Our objective in choosing $H_c(s)$ is that we would like the output $y(t)$ to follow the input $x(t)$. In particular, in addition to stabilizing the system, we would also like to design the system so that the error $e(t)$ decays to zero for certain specified inputs.

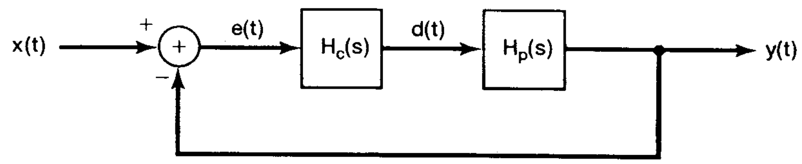


Figure P11.57

(a) Suppose that

$$H_p(s) = \frac{\alpha}{s+\alpha}, \quad \alpha \neq 0. \tag{P11.57–1}$$

Show that if $H_c(s) = K$ (which is known as *proportional* or *P* control), we can choose K so as to stabilize the system and so that $e(t) \to 0$ if $x(t) = \delta(t)$. Show that we *cannot* get $e(t) \to 0$ if $x(t) = u(t)$.

(b) Again let $H_p(s)$ be as in eq. (P11.57–1), and suppose that we use *proportional-plus-integral* (PI) control—that is,

$$H_c(s) = K_1 + \frac{K_2}{s}.$$

Show that we can choose K_1 and K_2 so as to stabilize the system, and we can also get $e(t) \to 0$ if $x(t) = u(t)$. Thus, the system can track a step. In fact, this illustrates a basic and important principle in feedback system design: To track a step $[X(s) = 1/s]$, we need an integrator $(1/s)$ in the feedback system. An extension of this principle is considered in the next problem.

(c) Suppose that

$$H_p(s) = \frac{1}{(s-1)^2}.$$

Show that we cannot stabilize this system with a PI controller, but that we can stabilize it and have it track a step if we use *proportional-plus-integral-plus-differential* (PID) control, i.e.,

$$H_c(s) = K_1 + \frac{K_2}{s} + K_3 s.$$

11.58. In Problem 11.57, we discussed how the presence of an integrator in a feedback system can make it possible for the system to track a step input with zero error in the steady state. In this problem, we extend the idea. Consider the feedback system depicted in Figure P11.58, and suppose that the overall closed-loop system is stable. Suppose also that

$$H(s) = \frac{K\prod_{k=1}^{m}(s-\beta_k)}{s^l\prod_{k=1}^{n-l}(s-\alpha_k)},$$

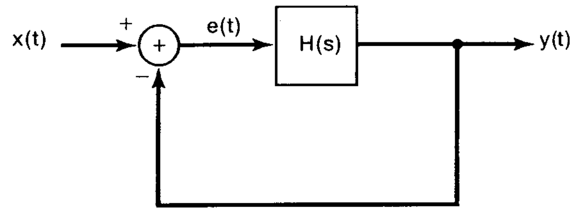


Figure P11.58

where the α_k and β_k are given nonzero numbers and l is a positive integer. The feedback system of Figure P11.58 is often referred to as a *Type l* feedback system.

(a) Use the final-value theorem (Section 9.5.10) to show that a Type 1 feedback system can track a step—that is, that

$$e(t) \to 0 \quad \text{if} \quad x(t) = u(t).$$

(b) Similarly, show that a Type 1 system cannot track a ramp, but rather, that

$$e(t) \to \text{a finite constant if} \quad x(t) = u_{-2}(t).$$

(c) Show that, for a Type 1 system, unbounded results ensue if

$$x(t) = u_{-k}(t)$$

with $k > 2$.

(d) More generally, show that, for a Type l system:

(i) $e(t) \to 0$ if $x(t) = u_{-k}(t)$ with $k \leq l$

(ii) $e(t) \to$ a finite constant if $x(t) = u_{(-l+1)}(t)$

(iii) $e(t) \to \infty$ if $x(t) = u_{-k}(t)$ with $k > l + 1$

11.59. (a) Consider the discrete-time feedback system of Figure P11.59. Suppose that

$$H(z) = \frac{1}{(z-1)(z+\frac{1}{2})}.$$

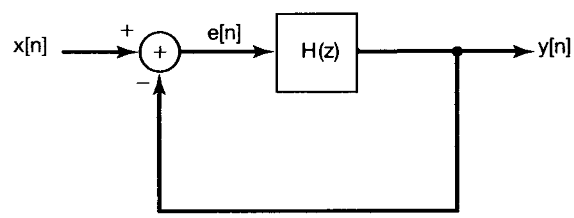


Figure P11.59

Show that this system can track a unit step in the sense that if $x[n] = u[n]$, then

$$\lim_{n \to \infty} e[n] = 0. \qquad \text{(P11.59–1)}$$

(b) More generally, consider the feedback system of Figure P11.59, and assume that the closed-loop system is stable. Suppose that $H(z)$ has a pole at $z = 1$.

Show that the system can track a unit step. [*Hint:* Express the transform $E(z)$ of $e[n]$ in terms of $H(z)$ and the transform of $u[n]$; explain why all the poles of $E(z)$ are inside the unit circle.]

(c) The results of parts (a) and (b) are discrete-time counterparts of the results for continuous-time systems discussed in Problems 11.57 and 11.58. In discrete time, we can also consider the design of the systems that track specified inputs *perfectly* after a finite number of steps. Such systems are known as *deadbeat feedback systems.*

Consider the discrete-time system of Figure P11.59 with

$$H(z) = \frac{z^{-1}}{1 - z^{-1}}.$$

Show that the overall closed-loop system is a deadbeat feedback system with the property that it tracks a step input exactly after one step: that is, if $x[n] = u[n]$, then $e[n] = 0,\ n \geq 1$.

(d) Show that the feedback system of Figure P11.59 with

$$H(z) = \frac{\frac{3}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1 + \frac{1}{4}z^{-1})(1 - z^{-1})}$$

is a deadbeat system with the property that the output tracks a unit step perfectly after a finite number of steps. At what time step does the error $e[n]$ first settle to zero?

(e) More generally, for the feedback system of Figure P11.59, find $H(z)$ so that $y[n]$ perfectly tracks a unit step for $n \geq N$ and, in fact, so that

$$e[n] = \sum_{k=0}^{N-1} a_k \delta[n - k], \qquad \text{(P11.59–2)}$$

where the a_k are specified constants: Hint: Use the relationship between $H(z)$ and $E(z)$ when the input is a unit step and $e[n]$ is given by eq. (P11.59–2).

(f) Consider the system of Figure P11.59 with

$$H(z) = \frac{z^{-1} + z^{-2} - z^{-3}}{(1 + z^{-1})(1 - z^{-1})^2}.$$

Show that this system tracks a ramp $x[n] = (n + 1)u[n]$ exactly after two time steps.

11.60. In this problem, we investigate some of the properties of sampled-data feedback systems and illustrate the use of such systems. Recall from Section 11.2.4 that in a sampled-data feedback system the output of a continuous-time system is sampled. The resulting sequence of samples is processed by a discrete-time system, the output of which is converted to a continuous-time signal that in turn is fed back and subtracted from the external input to produce the actual input to the continuous-time system.

(a) Consider the system within dashed lines in Figure 11.6(b). This is a discrete-time system with input $e[n]$ and output $p[n]$. Show that it is an LTI system. As we have indicated in the figure, we will let $F(z)$ denote the system function of this system.

(b) Show that in Figure 11.6(b) the discrete-time system with system function $F(z)$ is related to the continuous-time system with system function $H(s)$ by means of a *step-invariant* transformation. That is, if $s(t)$ is the step response of the continuous-time system and $q[n]$ is the step response of the discrete-time system, then

$$q[n] = s(nT) \qquad \text{for all } n.$$

(c) Suppose that

$$H(s) = \frac{1}{s-1}, \ \Re e\{s\} > 1.$$

Show that

$$F(z) = \frac{(e^T - 1)z^{-1}}{1 - e^T z^{-1}}, \quad |z| > e^T.$$

(d) Suppose that $H(s)$ is as in part (c) and that $G(z) = K$. Find the range of values of K for which the closed-loop discrete-time system of Figure 11.6(b) is stable.

(e) Suppose that

$$G(z) = \frac{K}{1 + \frac{1}{2}z^{-1}}.$$

Under what conditions on T can we find a value of K that stabilizes the overall system? Find a particular pair of values for K and T that yield a stable closed-loop system. *Hint:* Examine the root locus, and find the values for which the poles enter or leave the unit circle.