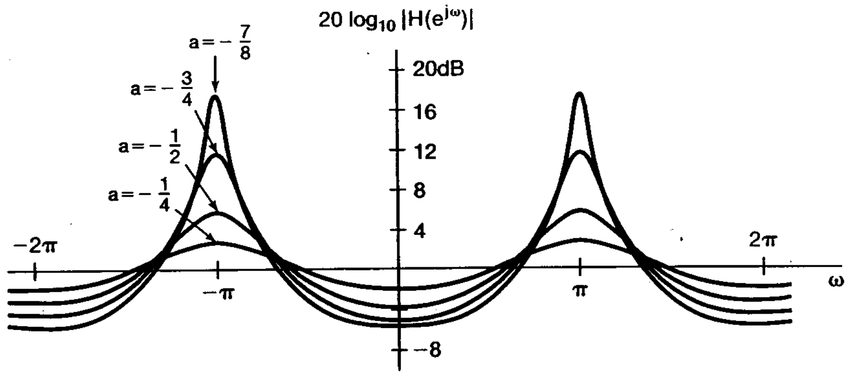
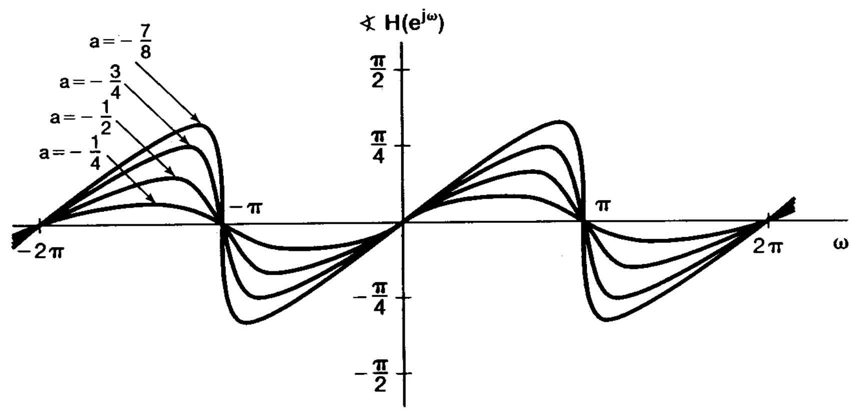




(b)

Figure 6.28 *Continued*

6.6.2 Second-Order Discrete-Time Systems

Consider next the second-order causal LTI system described by

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n], \tag{6.57}$$

with $0 < r < 1$ and $0 \le \theta \le \pi$. The frequency response for this system is

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}. \tag{6.58}$$

The denominator of $H(e^{j\omega})$ can be factored to obtain

$$H(e^{j\omega}) = \frac{1}{[1-(re^{j\theta})e^{-j\omega}][1-(re^{-j\theta})e^{-j\omega}]}. \tag{6.59}$$

For $\theta \neq 0$ or π, the two factors in the denominator of $H(e^{j\omega})$ are different, and a partial-fraction expansion yields

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}, \tag{6.60}$$

where

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \quad B = \frac{e^{-j\theta}}{2j\sin\theta}. \tag{6.61}$$

In this case, the impulse response of the system is

$$\begin{aligned} h[n] &= [A(re^{j\theta})^n + B(re^{-j\theta})^n]u[n] \\ &= r^n \frac{\sin[(n+1)\theta]}{\sin\theta} u[n]. \end{aligned} \tag{6.62}$$

For $\theta = 0$ or π, the two factors in the denominator of eq. (6.58) are the same. When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{(1-re^{-j\omega})^2} \tag{6.63}$$

and

$$h[n] = (n+1)r^n u[n]. \tag{6.64}$$

When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{(1+re^{-j\omega})^2} \tag{6.65}$$

and

$$h[n] = (n+1)(-r)^n u[n]. \tag{6.66}$$

The impulse responses for second-order systems are plotted in Figure 6.29 for a range of values of r and θ. From this figure and from eq. (6.62), we see that the rate of decay of $h[n]$ is controlled by r—i.e., the closer r is to 1, the slower is the decay in $h[n]$. Similarly, the value of θ determines the frequency of oscillation. For example, with $\theta = 0$ there is no oscillation in $h[n]$, while for $\theta = \pi$ the oscillations are rapid. The effect of different values of r and θ can also be seen by examining the step response of eq. (6.57). For $\theta \neq 0$ or π,

$$s[n] = h[n] * u[n] = \left[A\left(\frac{1-(re^{j\theta})^{n+1}}{1-re^{j\theta}}\right) + B\left(\frac{1-(re^{-j\theta})^{n+1}}{1-re^{-j\theta}}\right)\right]u[n]. \tag{6.67}$$

Also, using the result of Problem 2.52, we find that for $\theta = 0$,

$$s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2}r^n + \frac{r}{r-1}(n+1)r^n\right]u[n], \tag{6.68}$$

while for $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2}(-r)^n + \frac{r}{r+1}(n+1)(-r)^n\right]u[n]. \tag{6.69}$$

The step response is plotted in Figure 6.30, again for a range of values of r and θ.

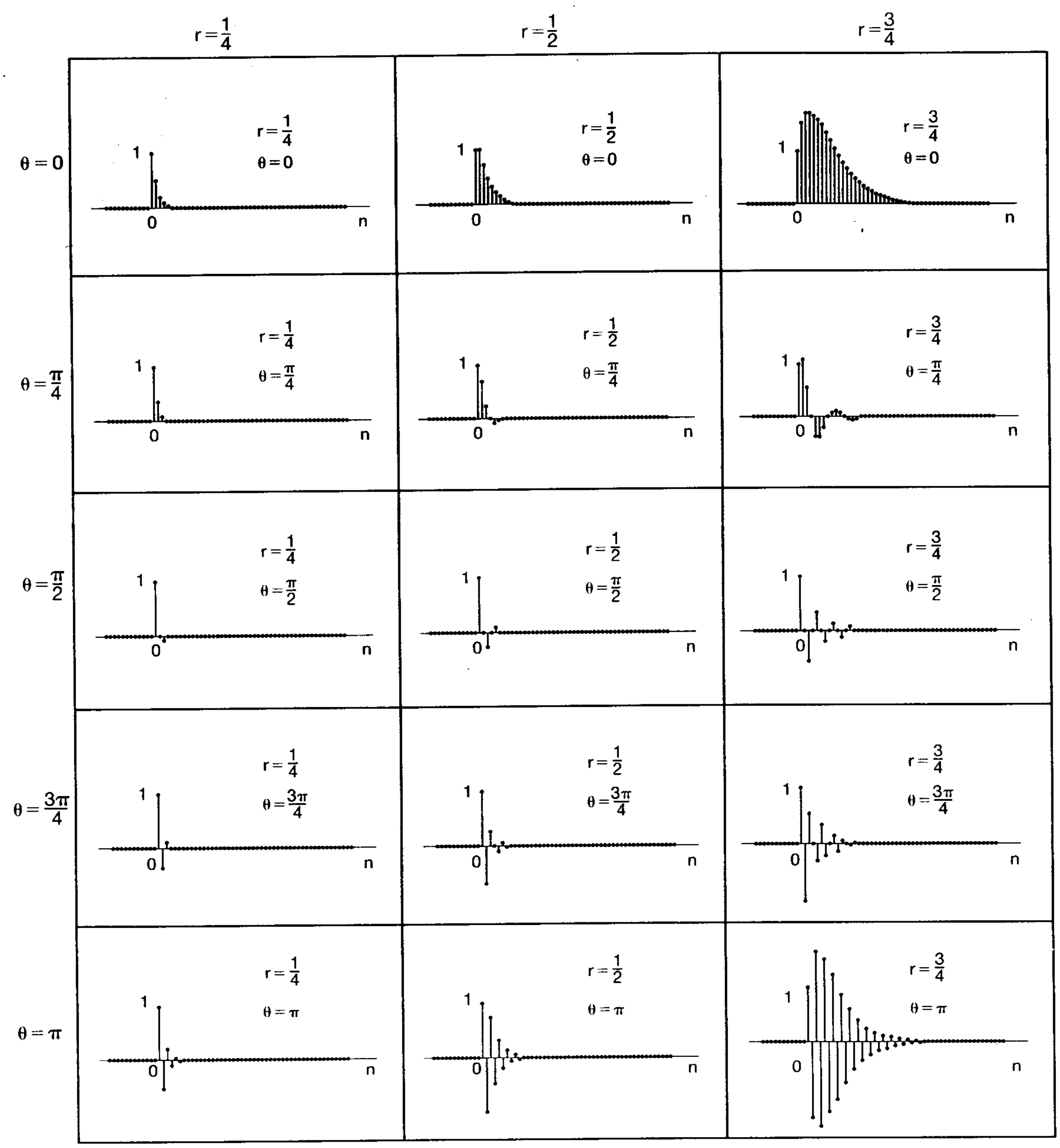


Figure 6.29 Impulse response of the second-order system of eq. (6.57) for a range of values of r and θ.

The second-order system given by eq. (6.57) is the counterpart of the *underdamped* second-order system in continuous time, while the special case of $\theta = 0$ is the critically damped case. That is, for any value of θ other than zero, the impulse response has a damped

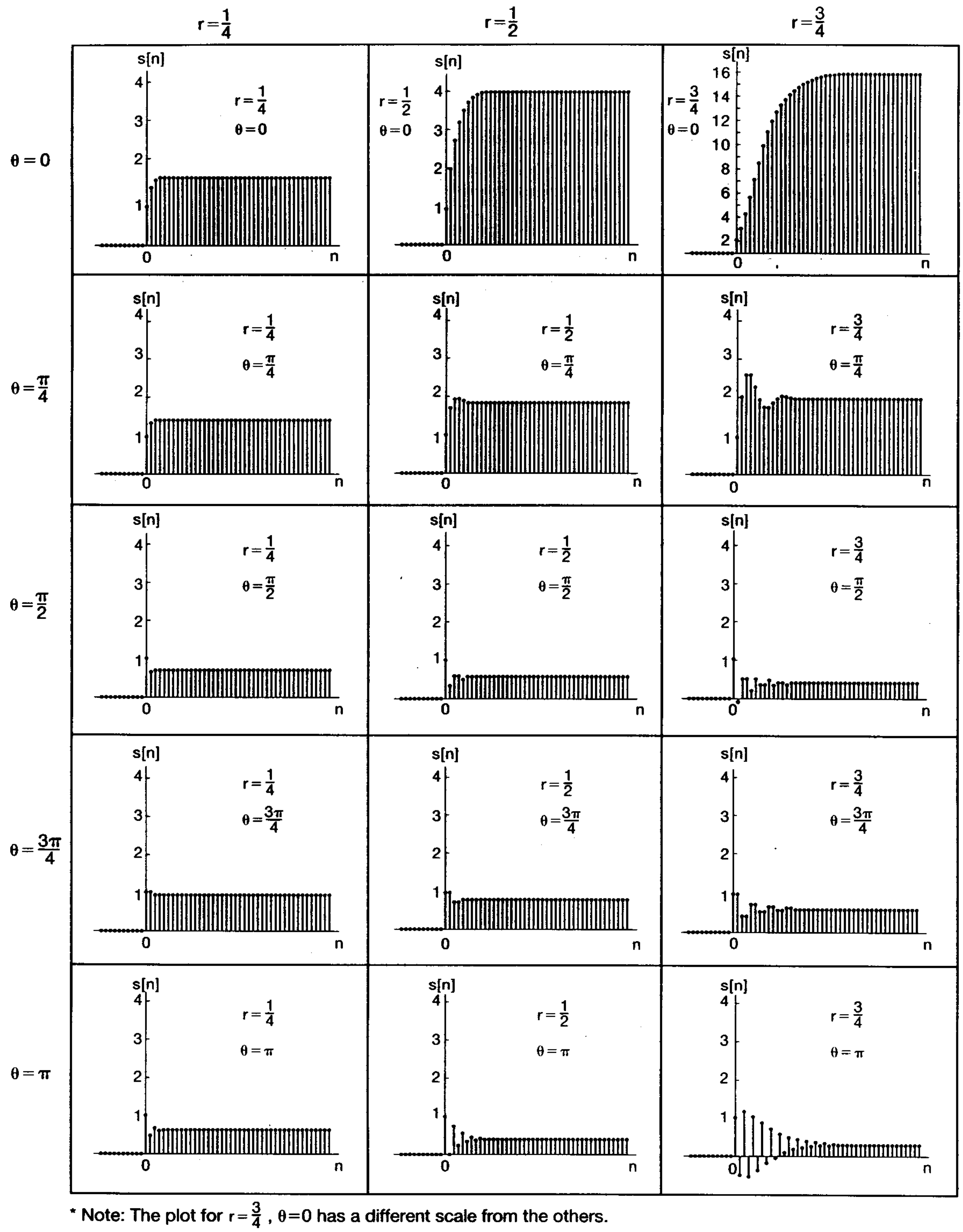


Figure 6.30 Step response of the second-order system of eq. (6.57) for a range of values of r and θ.

oscillatory behavior, and the step response exhibits ringing and overshoot. The frequency response of this system is depicted in Figure 6.31 for a number of values of r and θ. From Figure 6.31, we see that a band of frequencies is amplified, and r determines how sharply peaked the frequency response is within this band.

As we have just seen, the second-order system described in eq. (6.59) has factors with complex coefficients (unless $\theta = 0$ or π). It is also possible to consider second-order systems having factors with real coefficients. Specifically, consider

$$H(e^{j\omega}) = \frac{1}{(1 - d_1 e^{-j\omega})(1 - d_2 e^{-j\omega})}, \tag{6.70}$$

where d_1 and d_2 are both real numbers with $|d_1|, |d_2| < 1$. Equation (6.70) is the frequency response for the difference equation

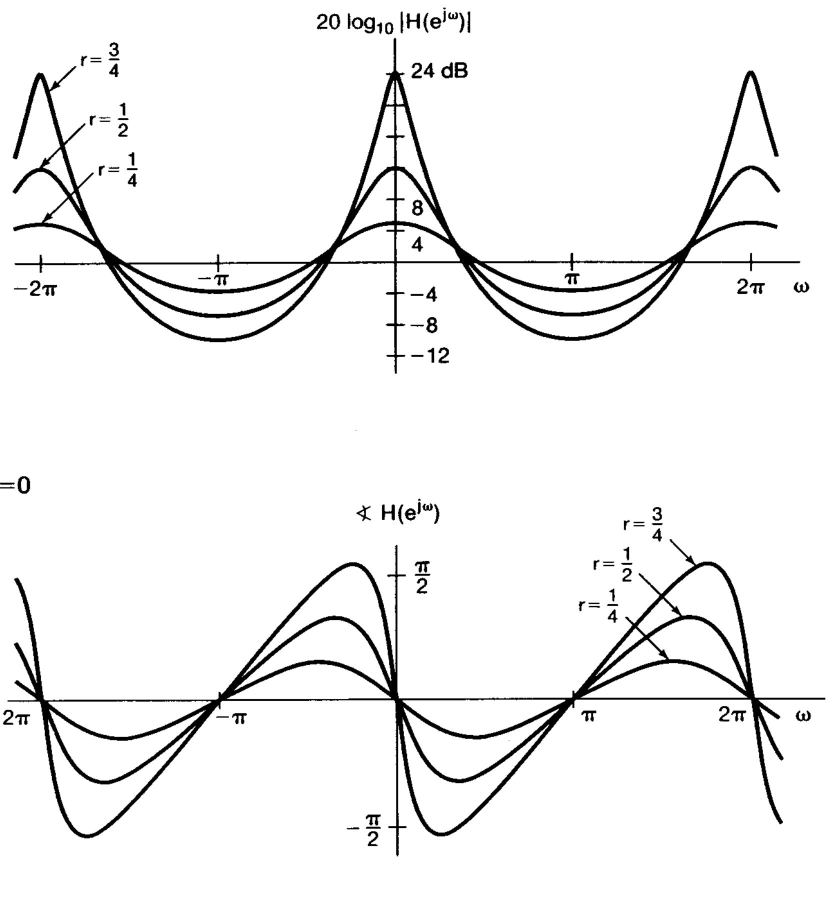
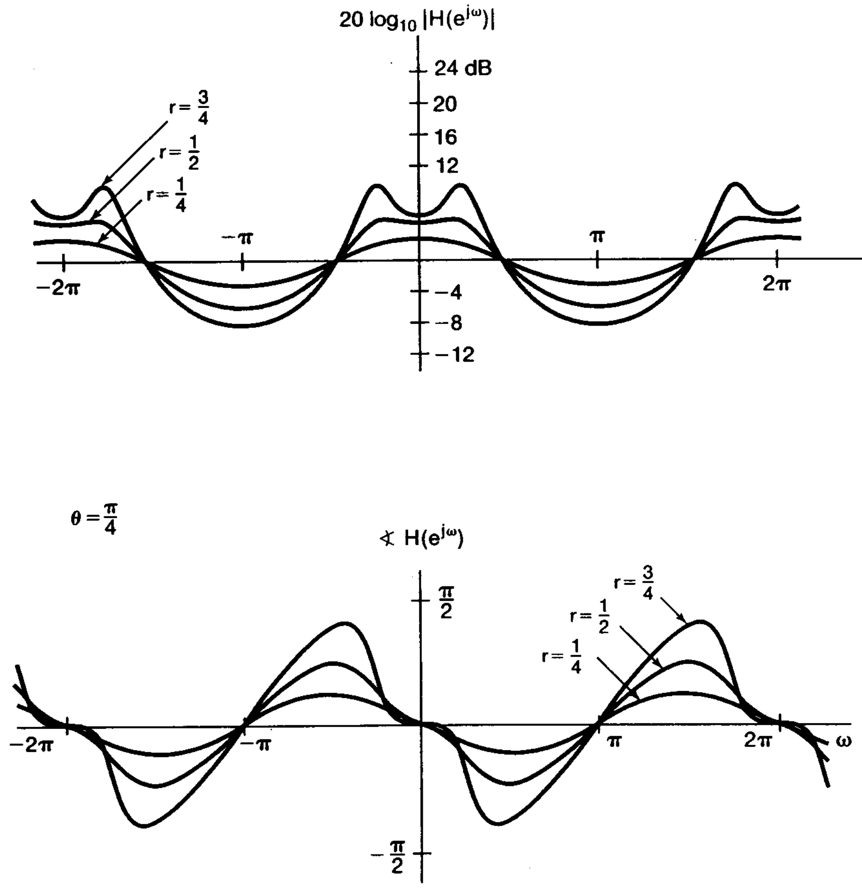


Figure 6.31 Magnitude and phase of the frequency response of the second-order system of eq. (6.57): (a) $\theta = 0$; (b) $\theta = \pi/4$; (c) $\theta = \pi/2$; (d) $\theta = 3\pi/4$; (e) $\theta = \pi$. Each plot contains curves corresponding to $r = 1/4$, $1/2$, and $3/4$.



(b)

Figure 6.31 *Continued*

$$y[n] - (d_1 + d_2)y[n-1] + d_1 d_2 y[n-2] = x[n]. \tag{6.71}$$

In this case,

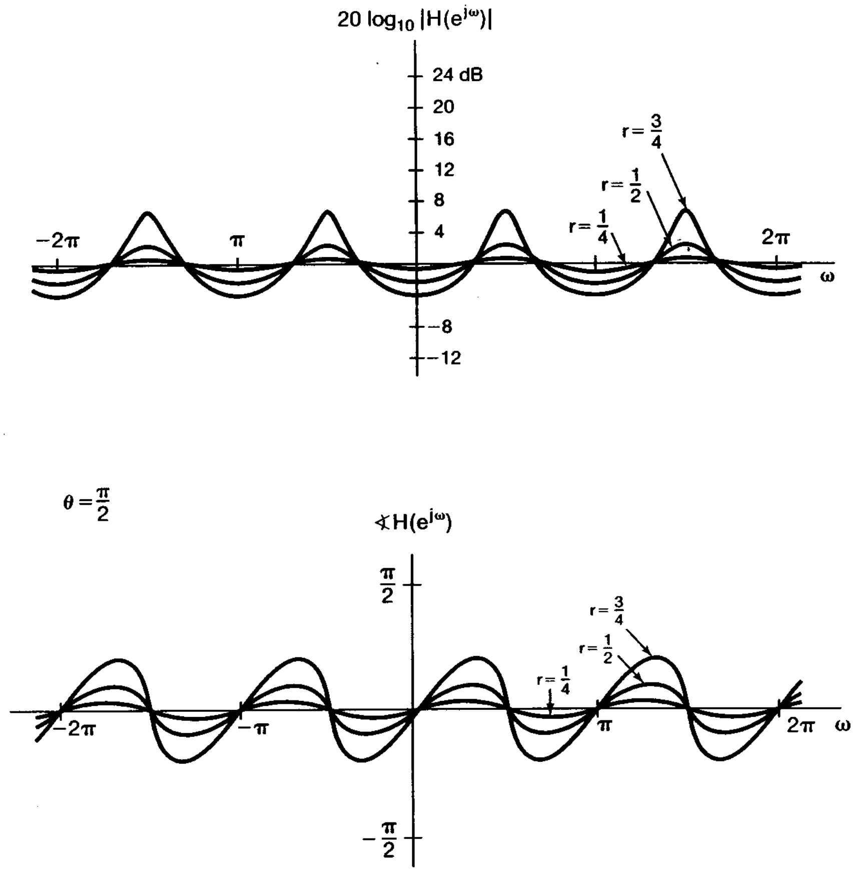
$$H(e^{j\omega}) = \frac{A}{1 - d_1 e^{-j\omega}} + \frac{B}{1 - d_2 e^{-j\omega}}, \tag{6.72}$$

where

$$A = \frac{d_1}{d_1 - d_2}, \quad B = \frac{d_2}{d_2 - d_1}. \tag{6.73}$$

Thus,

$$h[n] = [Ad_1^n + Bd_2^n]u[n], \tag{6.74}$$



(c)

Figure 6.31 *Continued*

which is the sum of two decaying real exponentials. Also,

$$s[n] = \left[A\left(\frac{1-d_1^{n+1}}{1-d_1}\right) + B\left(\frac{1-d_2^{n+1}}{1-d_2}\right)\right]u[n]. \tag{6.75}$$

The system with frequency response given by eq. (6.70) corresponds to the cascade of two first-order systems. Therefore, we can deduce most of its properties from our understanding of the first-order case. For example, the log-magnitude and phase plots for eq. (6.70) can be obtained by adding together the plots for each of the two first-order terms. Also, as we saw for first-order systems, the response of the system is fast if $|d_1|$ and $|d_2|$ are small, but the system has a long settling time if either of these magnitudes is near 1. Furthermore, if d_1 and d_2 are negative, the response is oscillatory. The case when both d_1 and d_2 are positive is the counterpart of the overdamped case in continuous time, with the impulse and step responses settling without oscillation.

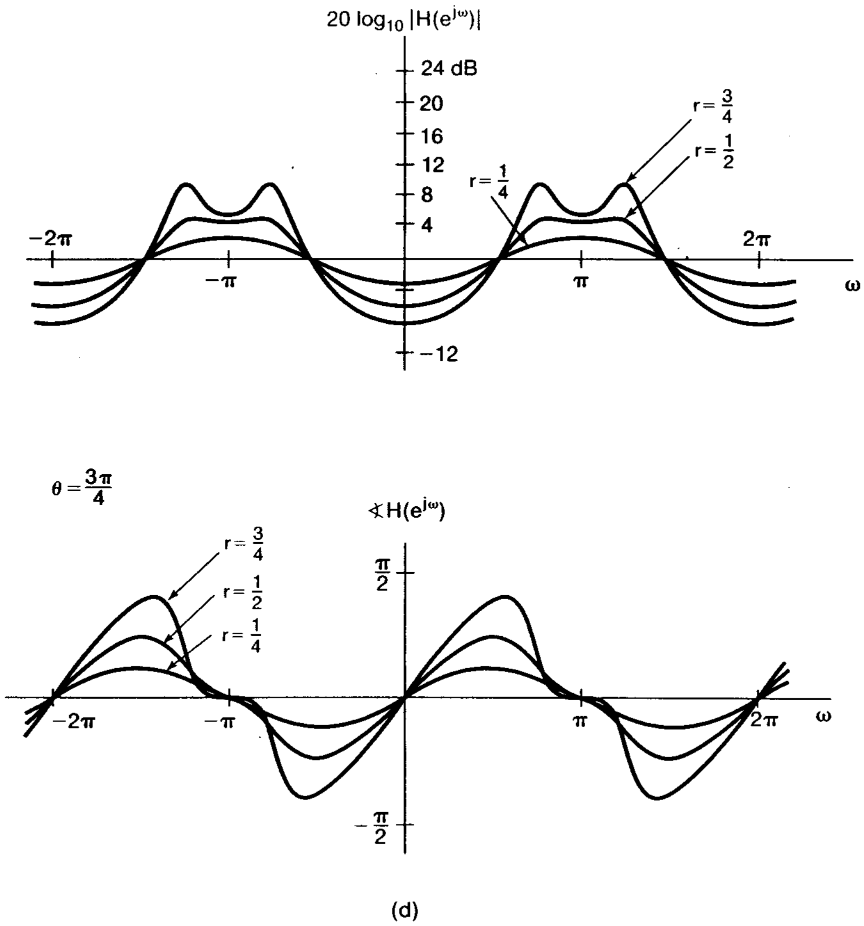
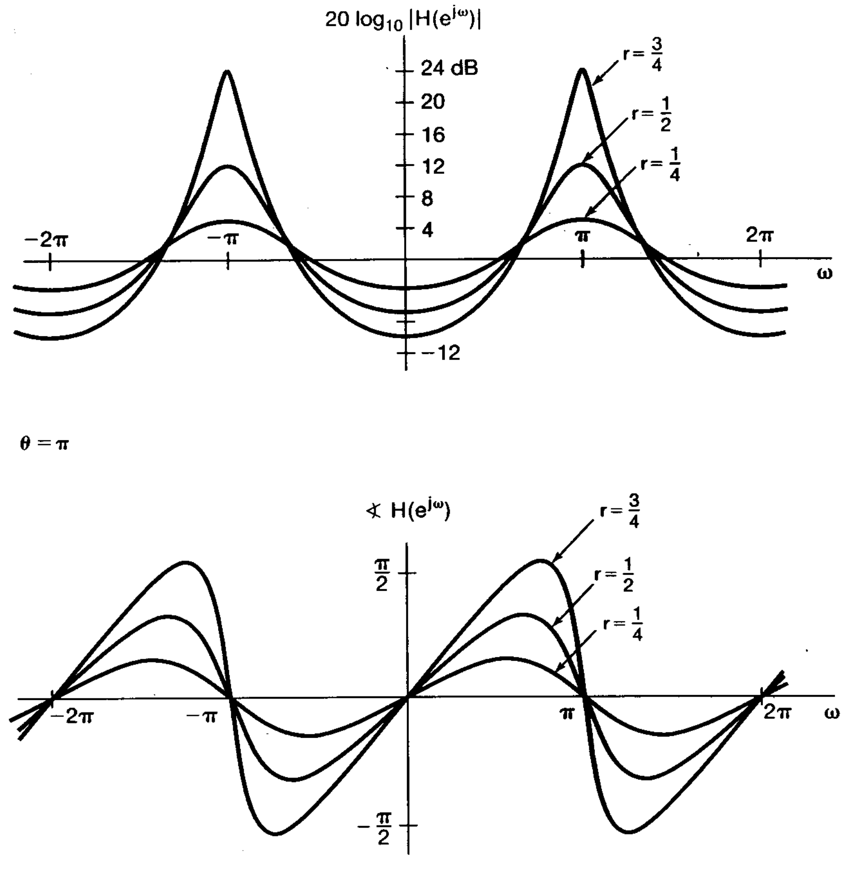


Figure 6.31 *Continued*

In this section, we have restricted attention to those causal first- and second-order systems that are stable and for which the frequency response can be defined. In particular, the causal system described by eq. (6.51) is unstable for $|a| \geq 1$. Also, the causal system described by eq. (6.56) is unstable if $r \geq 1$, and that described by eq. (6.71) is unstable if either $|d_1|$ or $|d_2|$ exceeds 1.

6.7 EXAMPLES OF TIME- AND FREQUENCY-DOMAIN ANALYSIS OF SYSTEMS

Throughout this chapter, we have illustrated the importance of viewing systems in both the time domain and the frequency domain and the importance of being aware of trade-offs in the behavior between the two domains. In this section, we illustrate some of these issues further. In Section 6.7.1, we discuss these trade-offs for continuous time in the context of an automobile suspension system. In Section 6.7.2, we discuss an important class of discrete-time filters referred to as moving-average or nonrecursive systems.



(e)

Figure 6.31 *Continued*

6.7.1 Analysis of an Automobile Suspension System

A number of the points that we have made concerning the characteristics and trade-offs in continuous-time systems can be illustrated in the interpretation of an automobile suspension system as a lowpass filter. Figure 6.32 shows a diagrammatic representation of a simple suspension system comprised of a spring and dashpot (shock absorber). The road surface can be thought of as a superposition of rapid small-amplitude changes in elevation (high frequencies), representing the roughness of the road surface, and gradual changes in elevation (low frequencies) due to the general topography. The automobile suspension system is generally intended to filter out rapid variations in the ride caused by the road surface (i.e., the system acts as a lowpass filter).

The basic purpose of the suspension system is to provide a smooth ride, and there is no sharp, natural division between the frequencies to be passed and those to be rejected. Thus, it is reasonable to accept and, in fact, prefer a lowpass filter that has a gradual

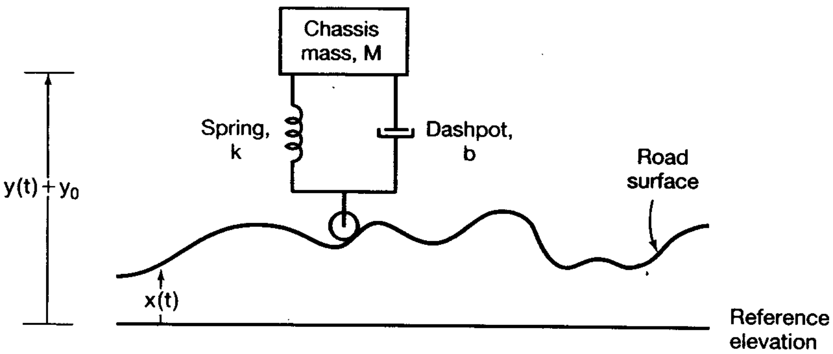


Figure 6.32 Diagrammatic representation of an automotive suspension system. Here, y_0 represents the distance between the chassis and the road surface when the automobile is at rest, $y(t) + y_0$ the position of the chassis above the reference elevation, and $x(t)$ the elevation of the road above the reference elevation.

transition from passband to stopband. Furthermore, the time-domain characteristics of the system are important. If the impulse response or step response of the suspension system exhibits ringing, then a large bump in the road (modeled as an impulse input) or a curb (modeled as a step input) will result in an uncomfortable oscillatory response. In fact, a common test for a suspension system is to introduce an excitation by depressing and then releasing the chassis. If the response exhibits ringing, it is an indication that the shock absorbers need to be replaced.

Cost and ease of implementation also play an important role in the design of automobile suspension systems. Many studies have been carried out to determine the most desirable frequency-response characteristics for suspension systems from the point of view of passenger comfort. In situations where the cost may be warranted, such as for passenger railway cars, intricate and costly suspension systems are used. For the automotive industry, cost is an important factor, and simple, less costly suspension systems are generally used. A typical automotive suspension system consists simply of the chassis connected to the wheels through a spring and a dashpot.

In the diagrammatic representation in Figure 6.32, y_0 represents the distance between the chassis and the road surface when the automobile is at rest, $y(t) + y_0$ the position of the chassis above the reference elevation, and $x(t)$ the elevation of the road above the reference elevation. The differential equation governing the motion of the chassis is then

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = kx(t) + b\frac{dx(t)}{dt}, \tag{6.76}$$

where M is the mass of the chassis and k and b are the spring and shock absorber constants, respectively. The frequency response of the system is

$$H(j\omega) = \frac{k + bj\omega}{(j\omega)^2M + b(j\omega) + k},$$

or

$$H(j\omega) = \frac{\omega_n^2 + 2\zeta\omega_n(j\omega)}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}, \tag{6.77}$$

where

$$\omega_n = \sqrt{\frac{k}{M}} \quad \text{and} \quad 2\zeta\omega_n = \frac{b}{M}.$$

As in Section 6.5.2, the parameter ω_n is referred to as the undamped natural frequency and ζ as the damping ratio. A Bode plot of the log magnitude of the frequency response in eq. (6.77) can be constructed by using first-order and second-order Bode plots. The Bode plot for eq. (6.77) is sketched in Figure 6.33 for several different values of the damping ratio. Figure 6.34 illustrates the step response for several different values of the damping ratio.

As we saw in Section 6.5.2, the filter cutoff frequency is controlled primarily through ω_n, or equivalently for a chassis with a fixed mass, by an appropriate choice of spring constant k. For a given ω_n, the damping ratio is then adjusted through the damping factor b associated with the shock asorbers. As the natural frequency ω_n is decreased, the suspension will tend to filter out slower road variations, thus providing a smoother ride. On the other hand, we see from Figure 6.34 that the rise time of the system increases, and thus the system will feel more sluggish. On the one hand, it would be desirable to keep ω_n small to improve the lowpass filtering; on the other hand, it would be desirable to have ω_n large for a rapid time response. These, of course, are conflicting requirements and illustrate the need for a trade-off between time-domain and frequency-domain characteristics. Typically, a suspension system with a low value of ω_n, so that the rise time is long, is characterized as "soft" and one with a high value of ω_n, so that the rise time is short, is characterized as "hard." From Figures 6.33 and 6.34, we observe also that, as the damping ratio decreases, the frequency response of the system cuts off more sharply, but the overshoot and ringing in the step response tend to increase, another trade-off between the time and frequency

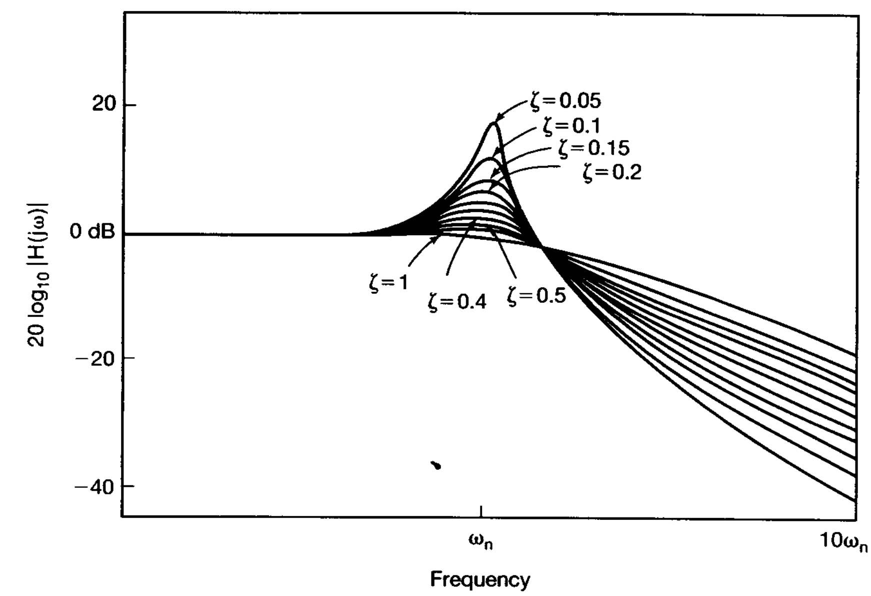


Figure 6.33 Bode plot for the magnitude of the frequency response of the automobile suspension system for several values of the damping ratio.

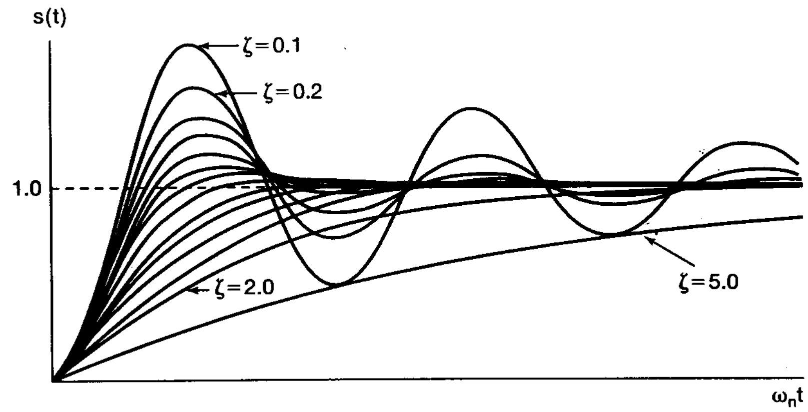


Figure 6.34 Step response of the automotive suspension system for various values of the damping ratio (ζ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0, 5.0).

domains. Generally, the shock absorber damping is chosen to have a rapid rise time and yet avoid overshoot and ringing. This choice corresponds to the critically damped case, with $\zeta = 1.0$, considered in Section 6.5.2.

6.7.2 Examples of Discrete-Time Nonrecursive Filters

In Section 3.11, we introduced the two basic classes of LTI filters described by difference equations, namely, recursive or infinite impulse response (IIR) filters and nonrecursive or finite impulse response (FIR) filters. Both of these classes of filters are of considerable importance in practice and have their own advantages and disadvantages. For example, recursive filters implemented as interconnections of the first- and second-order systems described in Section 6.6 provide a flexible class of filters that can be easily and efficiently implemented and whose characteristics can be adjusted by varying the number and the parameters of each of the component first- and second-order subsystems. On the other hand, as shown in Problem 6.64, it is not possible to design a causal, recursive filter with exactly linear phase, a property that we have seen is often desirable since, in that case, the effect of the phase on the output signal is a simple time delay. In contrast, as we show in this section, nonrecursive filters *can* have exactly linear phase. However, it is generally true that the same filter specifications require a higher order equation and hence more coefficients and delays when implemented with a nonrecursive equation, compared with a recursive difference equation. Consequently, for FIR filters, one of the principal trade-offs between the time and frequency domains is that increasing the flexibility in specifying the frequency domain characteristics of the filter, including, for example, achieving a higher degree of frequency selectivity, requires an FIR filter with an impulse response of longer duration.

One of the most basic nonrecursive filters, introduced in Section 3.11.2, is the moving-average filter. For this class of filters, the output is the average of the values of the input over a finite window:

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]. \tag{6.78}$$

The corresponding impulse response is a rectangular pulse, and the frequency response is

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}. \tag{6.79}$$

In Figure 6.35, we show the log magnitude for $M+N+1 = 33$ and $M+N+1 = 65$. The main, center lobe of each of these frequency responses corresponds to the effective passband of the corresponding filter. Note that, as the impulse response increases in length, the width of the main lobe of the magnitude of the frequency response decreases. This provides another example of the trade-off between the time and frequency domains. Specifically, in order to have a narrower passband, the filter in eqs. (6.78) and (6.79) must have a longer impulse response. Since the length of the impulse response of an FIR filter has a direct impact on the complexity of its implementation, this implies a trade-off between frequency selectivity and the complexity of the filter, a topic of central concern in filter design.

Moving-average filters are commonly applied in economic analysis in order to attenuate the short-term fluctuations in a variety of economic indicators in relation to longer term trends. In Figure 6.36, we illustrate the use of a moving-average filter of the form of eq. (6.78) on the weekly Dow Jones stock market index for a 10-year period. The weekly Dow Jones index is shown in Figure 6.36(a). Figure 6.36(b) is a 51-day moving average (i.e., $N = M = 25$) applied to that index, and Figure 6.36(c) is a 201-day moving average (i.e., $N = M = 100$) applied to the index. Both moving averages are considered useful, with the 51-day average tracking cyclical (i.e., periodic) trends that occur during the course of the year and the 201-day average primarily emphasizing trends over a longer time frame.

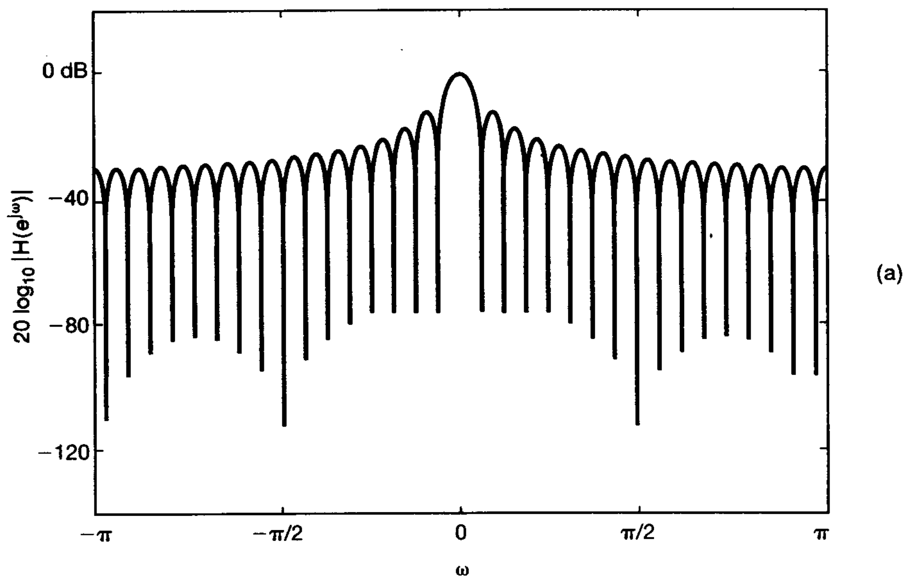
The more general form of a discrete-time nonrecursive filter is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k], \tag{6.80}$$

so that the output of this filter can be thought of as a weighted average of $N + M + 1$ neighboring points. The simple moving-average filter in eq. (6.78) then corresponds to setting all of these weights to the same value, namely, $1/(N+M+1)$. However, by choosing these coefficients in other ways, we have considerable flexibility in adjusting the filter's frequency response.

There are, in fact, a variety of techniques available for choosing the coefficients in eq. (6.80) so as to meet certain specifications on the filter, such as sharpening the transition band as much as possible for a filter of a given length (i.e., for N+M+1 fixed). These procedures are discussed in detail in a number of texts,[3] and although we do not discuss the procedures here, it is worth emphasizing that they rely heavily on the basic concepts and tools developed in this book. To illustrate how adjustment of the coefficients can influence

[3]See, for example, R. W. Hamming, *Digital Filters*, 3rd ed. (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1989); A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1989); and L. R. Rabiner and B. Gold, *Theory and Application of Digital Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975).



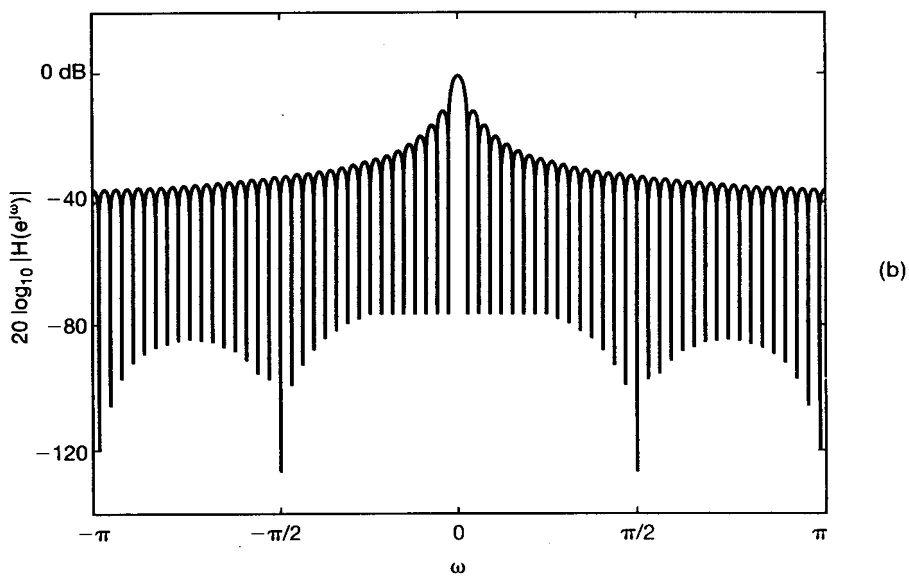


Figure 6.35 Log-magnitude plots for the moving-average filter of eqs. (6.78) and (6.79) for (a) $M + N + 1 = 33$ and (b) $M + N + 1 = 65$.

the response of the filter, let us consider a filter of the form of eq. (6.80), with $N = M = 16$ and the filter coefficients chosen to be

$$b_k = \begin{cases} \dfrac{\sin(2\pi k/33)}{\pi k}, & |k| \le 32 \\ 0, & |k| > 32 \end{cases}. \tag{6.81}$$

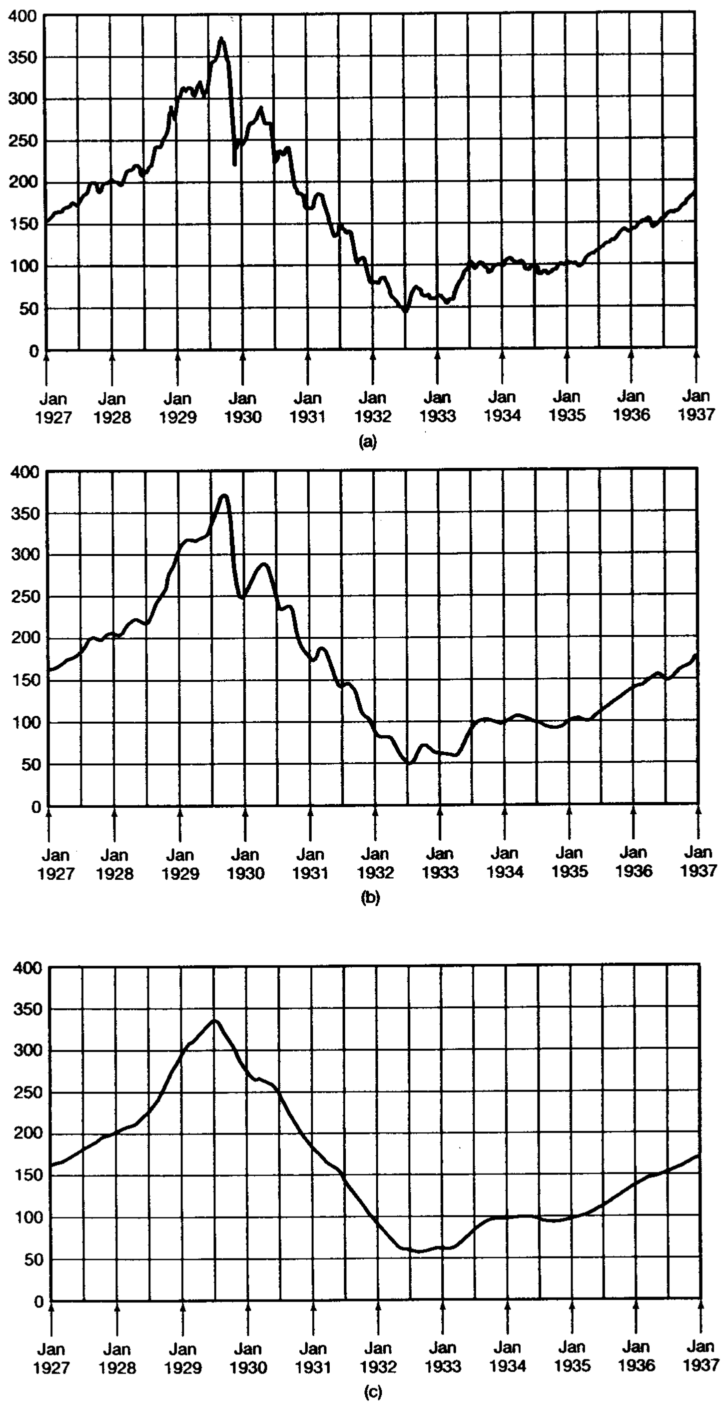


Figure 6.36 Effect of lowpass filtering on the Dow Jones weekly stock market index over a 10-year period using moving-average filters: (a) weekly index; (b) 51-day moving average applied to (a); (c) 201-day moving average applied to (a). The weekly stock market index and the two moving averages are discrete-time sequences. For clarity in the graphical display, the three sequences are shown here with their individual values connected by straight lines to form a continuous curve.

The impulse response of the filter is

$$h[n] = \begin{cases} \dfrac{\sin(2\pi n/33)}{\pi n}, & |n| \leq 32 \\ 0, & |n| > 32 \end{cases}. \tag{6.82}$$

Comparing this impulse response with eq. (6.20), we see that eq. (6.82) corresponds to truncating, for $|n| > 32$, the impulse response for the ideal lowpass filter with cutoff frequency $\omega_c = 2\pi/33$.

In general, the coefficients b_k can be adjusted so that the cutoff is at a desired frequency. For the example shown in Figure 6.37, the cutoff frequency was chosen to match approximately the cutoff frequency of Figure 6.35 for $N = M = 16$. Figure 6.37(a) shows the impulse response of the filter, and Figure 6.37(b) shows the log magnitude of the frequency response in dB. Comparing this frequency response to Figure 6.35, we observe that the passband of the filter has approximately the same width, but that the transition to

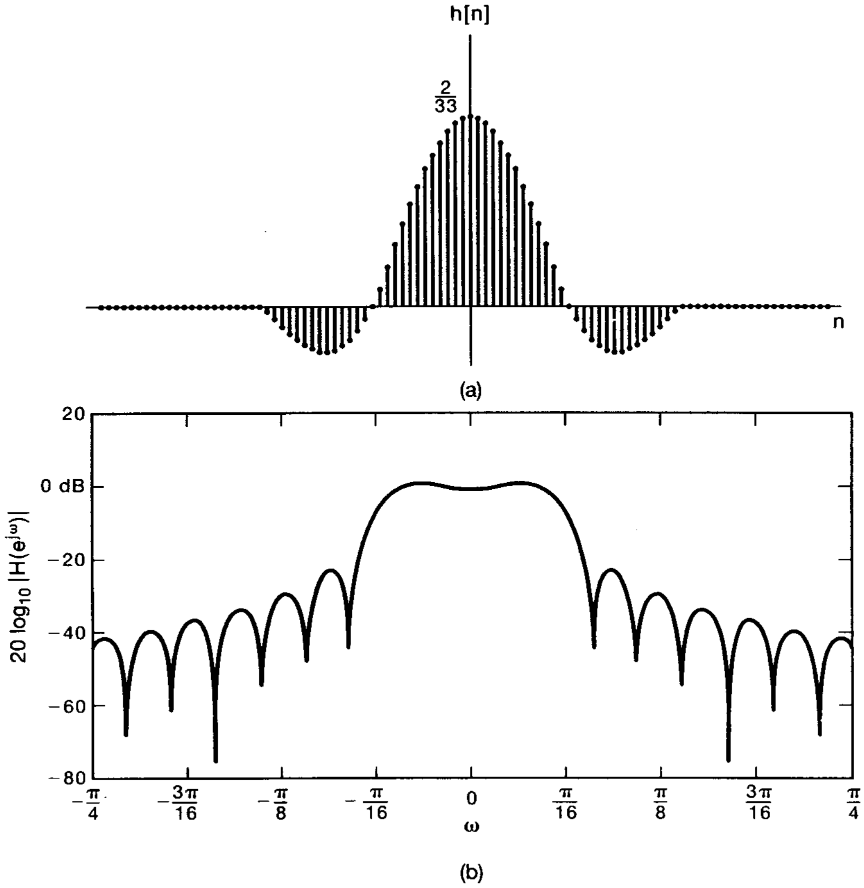


Figure 6.37 (a) Impulse response for the nonrecursive filter of eq. (6.82); (b) log magnitude of the frequency response of the filter.

the stopband is sharper. In Figures 6.38(a) and (b), the magnitudes (on a linear amplitude scale) of the two filters are shown for comparison. It should be clear from the comparison of the two examples that, by the intelligent choice of the weighting coefficients, the transition band can be sharpened. An example of a higher order lowpass filter ($N = M = 125$), with the coefficients determined through a numerical algorithm referred to as the Parks-McClellan algorithm,[4] is shown in Figure 6.39. This again illustrates the trade-off between the time and frequency domains: If we increase the length $N + M + 1$ of a filter, then, by a judicious choice of the filter coefficients in eq. (6.80), we can achieve sharper transition band behavior and a greater degree of frequency selectivity.

An important property of the examples we have given is that they all have zero or linear phase characteristics. For example, the phase of the moving-average filter of eq. (6.79) is $\omega[(N - M)/2]$. Also, since the impulse response in eq. (6.82) is real and even, the impulse response of the filter described by that equation is real and even, and thus has zero phase. From the symmetry properties of the Fourier transform of real signals, we know that *any* nonrecursive filter with an impulse response that is real and even will have a frequency response $H(e^{j\omega})$ that is real and even and, consequently, has zero phase. Such a filter, of course, is noncausal, since its impulse response $h[n]$ has nonzero values for $n < 0$. However, if a causal filter is required, then a simple change in the impulse response can achieve this, resulting in a system with *linear* phase. Specifically, since $h[n]$ is the impulse response of an FIR filter, it is identically zero outside a range of values centered at the origin

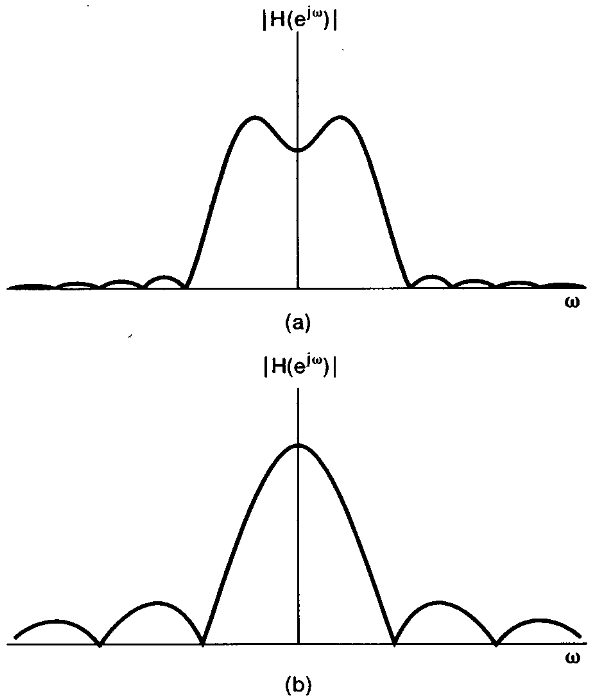


Figure 6.38 Comparison, on a linear amplitude scale, of the frequency responses of (a) Figure 6.37 and (b) Figure 6.35.

[4] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1989), Chap. 7.

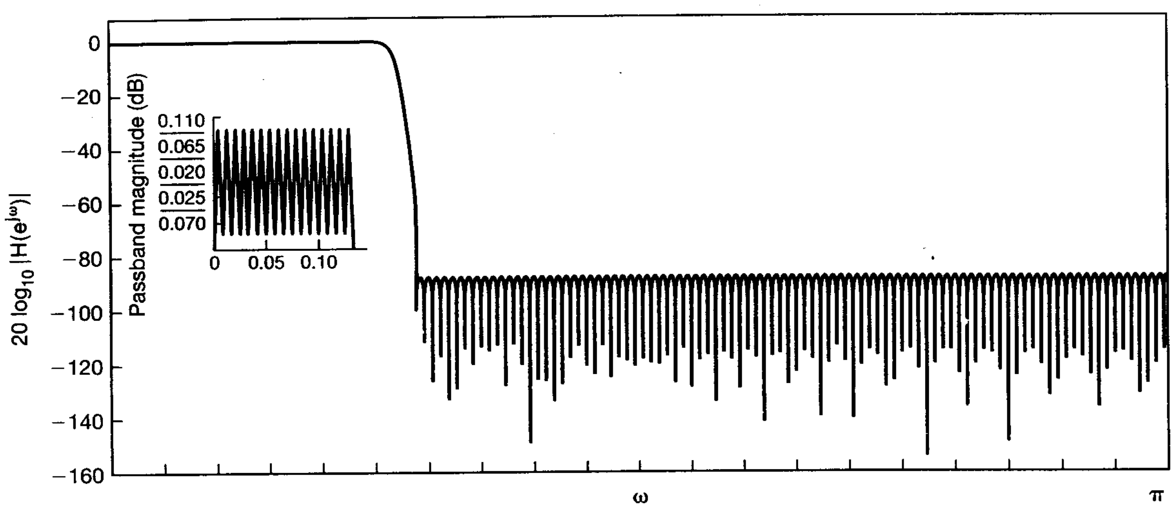


Figure 6.39 Lowpass nonrecursive filter with 251 coefficients designed to obtain the sharpest possible cutoff.

(i.e., $h[n] = 0$ for all $|n| > N$). If we now define the nonrecursive LTI system resulting from a simple N-step delay of $h[n]$, i.e.,

$$h_1[n] = h[n - N], \tag{6.83}$$

then $h_1[n] = 0$ for all $n < 0$, so that this LTI system is causal. Furthermore, from the time-shift property for discrete-time Fourier transforms, we see that the frequency response of the system is

$$H_1(e^{j\omega}) = H(e^{j\omega})e^{-j\omega N}. \tag{6.84}$$

Since $H(e^{j\omega})$ has zero phase, $H_1(e^{j\omega})$ does indeed have linear phase.

6.8 SUMMARY

In this chapter, we have built on the foundation of Fourier analysis of signals and systems developed in Chapters 3–5 in order to examine in more detail the characteristics of LTI systems and the effects they have on signals. In particular, we have taken a careful look at the magnitude and phase characteristics of signals and systems, and we have introduced log-magnitude and Bode plots for LTI systems. We have also discussed the impact of phase and phase distortion on signals and systems. This examination led us to understand the special role played by linear phase characteristics, which impart a constant delay at all frequencies and which, in turn, led to the concept of nonconstant group delay and dispersion associated with systems having nonlinear phase characteristics. Using these tools and insights, we took another look at frequency-selective filters and the time-frequency trade-offs involved. We examined the properties of both ideal and non-ideal frequency-selective filters and saw that time-frequency considerations, causality constraints, and implementation issues frequently make non-ideal filters, with transition bands and tolerance limits in the passbands and stopbands, the preferred choice.

We also examined in detail the time-frequency characteristics of first- and second-order systems in both continuous and discrete time. We noted in particular the trade-off between the response time of these systems and the frequency-domain bandwidth. Since first- and second-order systems are the building blocks for more complex, higher order LTI systems, the insights developed for those basic systems are of considerable use in practice.

Finally, we presented several examples of LTI systems in order to illustrate many of the points developed in the chapter. In particular, we examined a simple model of an automobile suspension system to provide a concrete example of the time-response–frequency-response concerns that drive system design in practice. We also considered several examples of discrete-time nonrecursive filters, ranging from simple moving-average filters to higher order FIR filters designed to have enhanced frequency selectivity. We saw, in addition, that FIR filters can be designed so as to have exactly linear phase. These examples, the development of the tools of Fourier analysis that preceded them, and the insights those tools provide illustrate the considerable value of the methods of Fourier analysis in analyzing and designing LTI systems.

Chapter 6 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining two sections contain problems belonging to the basic and advanced categories, respectively.

BASIC PROBLEMS WITH ANSWERS

6.1. Consider a continuous-time LTI system with frequency response $H(j\omega) = |H(j\omega)|\, e^{j\sphericalangle H(j\omega)}$ and real impulse response $h(t)$. Suppose that we apply an input $x(t) = \cos(\omega_0 t + \phi_0)$ to this system. The resulting output can be shown to be of the form

$$y(t) = Ax(t - t_0),$$

where A is a nonnegative real number representing an *amplitude-scaling* factor and t_0 is a time delay.

(a) Express A in terms of $|H(j\omega_0)|$.

(b) Express t_0 in terms of $\sphericalangle H(j\omega_0)$.

6.2. Consider a discrete-time LTI system with frequency response $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\sphericalangle H(e^{j\omega})}$ and real impulse response $h[n]$. Suppose that we apply the input $x[n] = \sin(\omega_0 n + \phi_0)$ to this system. The resulting output can be shown to be of the form

$$y[n] = |H(e^{j\omega_0})|x[n - n_0],$$

provided that $\sphericalangle H(e^{j\omega_0})$ and ω_0 are related in a particular way. Determine this relationship.

6.3. Consider the following frequency response for a causal and stable LTI system:

$$H(j\omega) = \frac{1 - j\omega}{1 + j\omega}.$$

(a) Show that $|H(j\omega)| = A$, and determine the value of A.

(b) Determine which of the following statements is true about $\tau(\omega)$, the group delay of the system. (*Note:* $\tau(\omega) = -d(\sphericalangle H(j\omega))/d\omega$, where $\sphericalangle H(j\omega)$ is expressed in a form that does not contain any discontinuities.)

1. $\tau(\omega) = 0$ for $\omega > 0$
2. $\tau(\omega) > 0$ for $\omega > 0$
3. $\tau(\omega) < 0$ for $\omega > 0$

6.4. Consider a linear-phase discrete-time LTI system with frequency response $H(e^{j\omega})$ and real impulse response $h[n]$. The group delay function for such a system is defined as

$$\tau(\omega) = -\frac{d}{d\omega}\sphericalangle H(e^{j\omega}),$$

where $\sphericalangle H(e^{j\omega})$ has no discontinuities. Suppose that, for this system,

$$|H(e^{j\pi/2})| = 2, \quad \sphericalangle H(e^{j0}) = 0, \quad \text{and} \quad \tau\left(\frac{\pi}{2}\right) = 2.$$

Determine the output of the system for each of the following inputs:

(a) $\cos(\frac{\pi}{2}n)$ **(b)** $\sin(\frac{7\pi}{2}n + \frac{\pi}{4})$

6.5. Consider a continuous-time ideal bandpass filter whose frequency response is

$$H(j\omega) = \begin{cases} 1, & \omega_c \le |\omega| \le 3\omega_c \\ 0, & \text{elsewhere} \end{cases}.$$

(a) If $h(t)$ is the impulse response of this filter, determine a function $g(t)$ such that

$$h(t) = \left(\frac{\sin \omega_c t}{\pi t}\right) g(t).$$

(b) As ω_c is increased, does the impulse response of the filter get more concentrated or less concentrated about the origin?

6.6. Consider a discrete-time ideal highpass filter whose frequency response is specified as

$$H(e^{j\omega}) = \begin{cases} 1, & \pi - \omega_c \le |\omega| \le \pi \\ 0, & |\omega| < \pi - \omega_c \end{cases}$$

(a) If $h[n]$ is the impulse response of this filter, determine a function $g[n]$ such that

$$h[n] = \left(\frac{\sin \omega_c n}{\pi n}\right) g[n].$$

(b) As ω_c is increased, does the impulse response of the filter get more concentrated or less concentrated about the origin?

6.7. A continuous-time lowpass filter has been designed with a passband frequency of 1,000 Hz, a stopband frequency of 1,200 Hz, passband ripple of 0.1, and stopband ripple of 0.05. Let the impulse response of this lowpass filter be denoted by $h(t)$. We wish to convert the filter into a bandpass filter with impulse response

$$g(t) = 2h(t)\cos(4{,}000\pi t).$$

Assuming that $|H(j\omega)|$ is negligible for $|\omega| > 4{,}000\pi$, answer the following questions:

(a) If the passband ripple for the bandpass filter is constrained to be 0.1, what are the two passband frequencies associated with the bandpass filter?

(b) If the stopband ripple for the bandpass filter is constrained to be 0.05, what are the two stopband frequencies associated with the bandpass filter?

6.8. A causal, nonideal lowpass filter is designed with frequency response $H(e^{j\omega})$. The difference equation relating the input $x[n]$ and output $y[n]$ for this filter is specified as

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

The filter also satisfies the following specifications for the magnitude of its frequency response:

$$\begin{aligned} \text{passband frequency} &= \omega_p, \\ \text{passband tolerance} &= \delta_p, \\ \text{stopband frequency} &= \omega_s, \\ \text{stopband tolerance} &= \delta_s. \end{aligned}$$

Now consider a causal LTI system whose input and output are related by the difference equation

$$y[n] = \sum_{k=1}^{N} (-1)^k a_k y[n-k] + \sum_{k=0}^{M} (-1)^k b_k x[n-k].$$

Show that this filter has a passband with a tolerance of δ_p, and specify the corresponding location of the passband.

6.9. Consider a continuous-time causal and stable LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{dy(t)}{dt} + 5y(t) = 2x(t).$$

What is the final value $s(\infty)$ of the step response $s(t)$ of this filter? Also, determine the value of t_0 for which

$$s(t_0) = s(\infty)\left[1 - \frac{1}{e^2}\right].$$

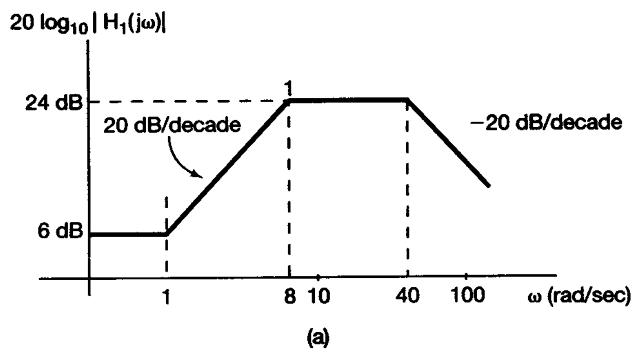
6.10. For each first-order system whose frequency response is as follows, specify the straight-line approximation of the Bode magnitude plot:

(a) $40\left(\frac{j\omega+0.1}{j\omega+40}\right)$ **(b)** $0.04\left(\frac{j\omega+50}{j\omega+0.2}\right)$

6.11. For each second-order system whose frequency response is as follows, specify the straight-line approximation of the Bode magnitude plot:

(a) $\frac{250}{(j\omega)^2+50.5j\omega+25}$ **(b)** $0.02\frac{j\omega+50}{(j\omega)^2+0.2j\omega+1}$

6.12. A continuous-time LTI system S with frequency response $H(j\omega)$ is constructed by cascading two continuous-time LTI systems with frequency responses $H_1(j\omega)$ and $H_2(j\omega)$, respectively. Figures P6.12(a) and P6.12(b) show the straight-line approximations of the Bode magnitude plots of $H_1(j\omega)$ and $H(j\omega)$, respectively. Specify $H_2(j\omega)$.



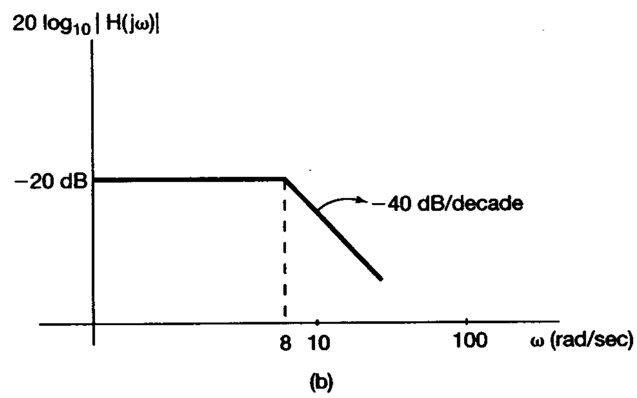


Figure P6.12

6.13. The straight-line approximation of the Bode magnitude plot of a second-order continuous-time LTI system S is shown in Figure P6.13. S may be constructed by

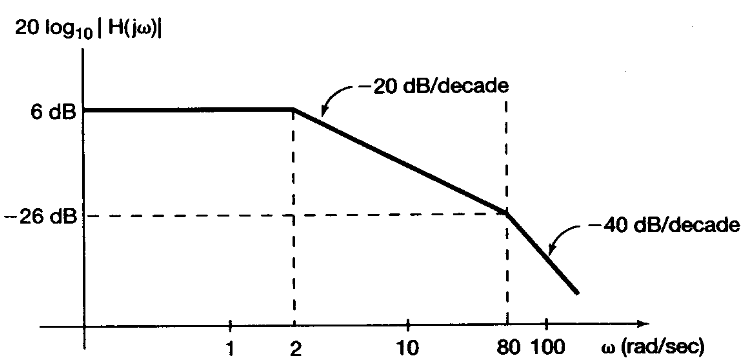


Figure P6.13

either connecting two first-order systems S_1 and S_2 in cascade or two first-order systems S_3 and S_4 in parallel. Determine which, if any, of the following statements are true or false. Justify your answers.

(a) The frequency responses of S_1 and S_2 may be determined uniquely.

(b) The frequency responses of S_3 and S_4 may be determined uniquely.

6.14. The straight-line approximation of the Bode magnitude plot of a causal and stable continuous-time LTI system S is shown in Figure P6.14. Specify the frequency response of a system that is the inverse of S.

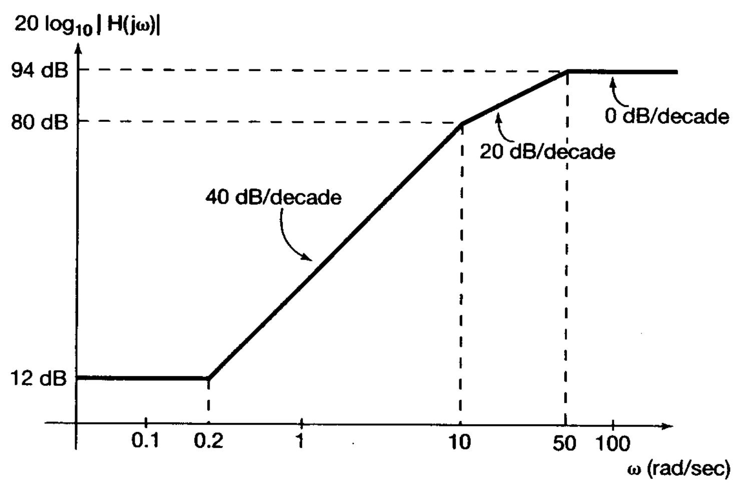


Figure P6.14

6.15. For each of the following second-order differential equations for causal and stable LTI systems, determine whether the corresponding impulse response is underdamped, overdamped, or critically damped:

(a) $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = x(t)$

(b) $5\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = 7x(t)$

(c) $\frac{d^2y(t)}{dt^2} + 20\frac{dy(t)}{dt} + y(t) = x(t)$

(d) $5\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 5y(t) = 7x(t) + \frac{1}{3}\frac{dx(t)}{dt}$

6.16. A particular first-order causal and stable discrete-time LTI system has a step response whose maximum overshoot is 50% of its final value. If the final value is 1, determine a difference equation relating the input $x[n]$ and output $y[n]$ of this filter.

6.17. For each of the following second-order difference equations for causal and stable LTI systems, determine whether or not the step response of the system is oscillatory:

(a) $y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n]$

(b) $y[n] - y[n-1] + \frac{1}{4}y[n-2] = x[n]$

6.18. Consider the continuous-time LTI system implemented as the RC circuit shown in Figure P6.18. The voltage source $x(t)$ is considered the input to this system. The voltage $y(t)$ across the capacitor is considered the system output. Is it possible for the step response of the system to have an oscillatory behavior?

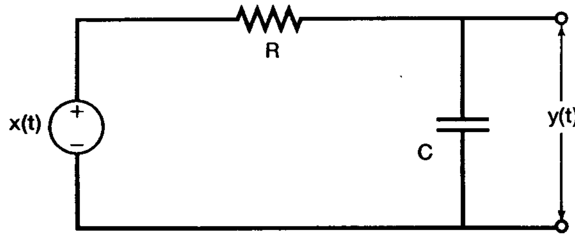


Figure P6.18

6.19. Consider the continuous-time LTI system implemented as the *RLC* circuit shown in Figure P6.19. The voltage source $x(t)$ is considered the input to this system. The voltage $y(t)$ across the capacitor is considered the system output. How should R, L, and C be related so that there is no oscillation in the step response?

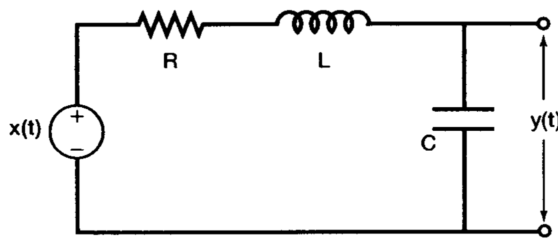


Figure P6.19

6.20. Consider a nonrecursive filter with the impulse response shown in Figure P6.20. What is the group delay as a function of frequency for this filter?

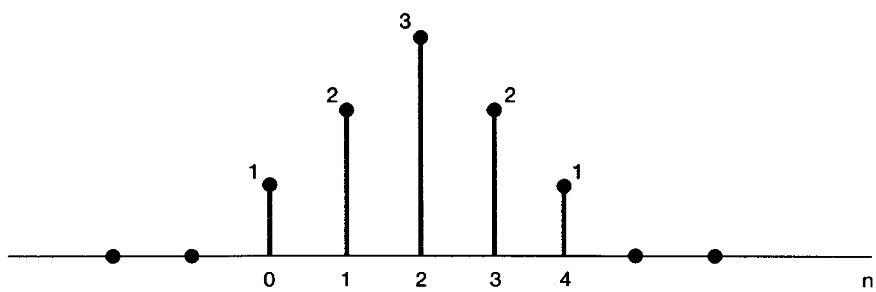


Figure P6.20

BASIC PROBLEMS

6.21. A causal LTI filter has the frequency response $H(j\omega)$ shown in Figure P6.21. For each of the input signals given below, determine the filtered output signal $y(t)$.

(a) $x(t) = e^{jt}$ **(b)** $x(t) = (\sin \omega_0 t)u(t)$

(c) $X(j\omega) = \frac{1}{(j\omega)(6+j\omega)}$ **(d)** $X(j\omega) = \frac{1}{2+j\omega}$

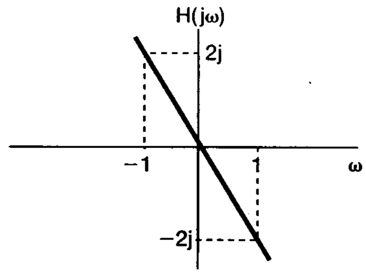


Figure P6.21

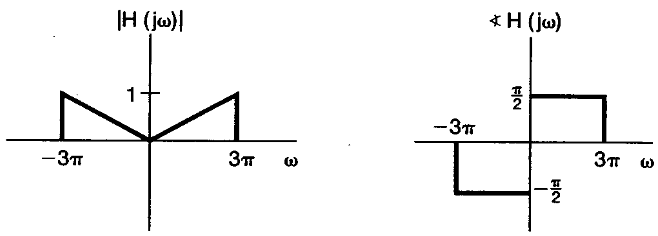
6.22. Shown in Figure P6.22(a) is the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For each of the input signals $x(t)$ below, determine the filtered output signal $y(t)$.

(a) $x(t) = \cos(2\pi t + \theta)$

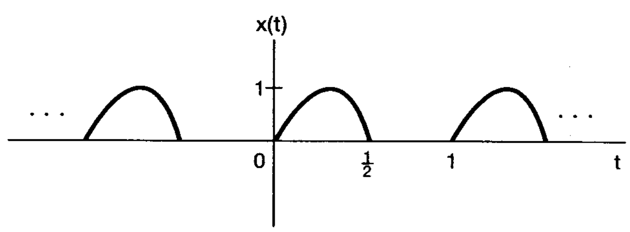
(b) $x(t) = \cos(4\pi t + \theta)$

(c) $x(t)$ is a half-wave rectified sine wave of period, as sketched in Figure P6.22(b).

$$x(t) = \begin{cases} \sin 2\pi t, & m \le t \le (m + \frac{1}{2}) \\ 0, & (m + \frac{1}{2}) \le t \le m \text{ for any integer } m \end{cases}$$



(a)



(b) Figure P6.22

6.23. Shown in Figure P6.23 is $|H(j\omega)|$ for a lowpass filter. Determine and sketch the impulse response of the filter for each of the following phase characteristics:

(a) $\sphericalangle H(j\omega) = 0$

(b) $\sphericalangle H(j\omega) = \omega T$, where T is a constant

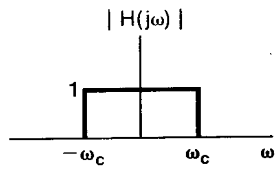


Figure P6.23

(c) $\sphericalangle H(j\omega) = \begin{cases} \frac{\pi}{2}, & \omega > 0 \\ \frac{-\pi}{2}, & \omega < 0 \end{cases}$

6.24. Consider a continuous-time lowpass filter whose impulse response $h(t)$ is known to be real and whose frequency response magnitude is given as:

$$|H(j\omega)| = \begin{cases} 1, & |\omega| \le 200\pi \\ 0, & \text{otherwise} \end{cases}.$$

(a) Determine and sketch the real-valued impulse response $h(t)$ for this filter when the corresponding group delay function is specified as:
(i) $\tau(\omega) = 5$ **(ii)** $\tau(\omega) = \frac{5}{2}$ **(iii)** $\tau(\omega) = -\frac{5}{2}$

(b) If the impulse response $h(t)$ had not been specified to be real, would knowledge of $|H(j\omega)|$ and $\tau(\omega)$ be sufficient to determine $h(t)$ uniquely? Justify your answer.

6.25. By computing the group delay at two selected frequencies, verify that each of the following frequency responses has nonlinear phase.
(a) $H(j\omega) = \frac{1}{j\omega+1}$ **(b)** $H(j\omega) = \frac{1}{(j\omega+1)^2}$ **(c)** $H(j\omega) = \frac{1}{(j\omega+1)(j\omega+2)}$

6.26. Consider an ideal highpass filter whose frequency response is specified as

$$H(j\omega) = \begin{cases} 1, & |\omega| > \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

(a) Determine the impulse response $h(t)$ for this filter.
(b) As ω_c is increased, does $h(t)$ get more or less concentrated about the origin?
(c) Determine $s(0)$ and $s(\infty)$, where $s(t)$ is the step response of the filter.

6.27. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(a) Determine the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

of the system, and sketch its Bode plot.
(b) Specify, as a function of frequency, the group delay associated with this system.
(c) If $x(t) = e^{-t}u(t)$, determine $Y(j\omega)$, the Fourier transform of the output.

(d) Using the technique of partial-fraction expansion, determine the output $y(t)$ for the input $x(t)$ in part (c).

(e) Repeat parts (c) and (d), first if the input has as its Fourier transform

(i) $X(j\omega) = \frac{1+j\omega}{2+j\omega}$,

then if

(ii) $X(j\omega) = \frac{2+j\omega}{1+j\omega}$,

and finally, if

(iii) $X(j\omega) = \frac{1}{(2+j\omega)(1+j\omega)}$

6.28. **(a)** Sketch the Bode plots for the following frequency responses:

(i) $1 + (j\omega/10)$ **(ii)** $1 - (j\omega/10)$

(iii) $\frac{16}{(j\omega+2)^4}$ **(iv)** $\frac{1-(j\omega/10)}{1+j\omega}$

(v) $\frac{(j\omega/10)-1}{1+j\omega}$ **(vi)** $\frac{1+(j\omega/10)}{1+j\omega}$

(vii) $\frac{1-(j\omega/10)}{(j\omega)^2+(j\omega)+1}$ **(viii)** $\frac{10+5j\omega+10(j\omega)^2}{1+(j\omega/10)}$

(ix) $1 + j\omega + (j\omega)^2$ **(x)** $1 - j\omega + (j\omega)^2$

(xi) $\frac{(j\omega+10)(10j\omega+1)}{[(j\omega/100+1)][((j\omega)^2+j\omega+1)]}$

(b) Determine and sketch the impulse response and the step response for the system with frequency response (iv). Do the same for the system with frequency response (vi).

The system given in (iv) is often referred to as a non-minimum-phase system, while the system specified in (vi) is referred to as being a minimum phase. The corresponding impulse responses of (iv) and (vi) are referred to as a non-minimum-phase signal and a minimum-phase signal, respectively. By comparing the Bode plots of these two frequency responses, we can see that they have identical magnitudes; however, the magnitude of the *phase* of the system of (iv) is larger than for the system of (vi).

We can also note differences in the time-domain behavior of the two systems. For example, the impulse response of the minimum-phase system has more of its energy concentrated near $t = 0$ than does the impulse response of the non-minimum-phase system. In addition, the step response of (iv) initially has the opposite sign from its asymptotic value as $t \to \infty$, while this is not the case for the system of (vi).

The important concept of minimum- and non-minimum-phase systems can be extended to more general LTI systems than the simple first-order systems we have treated here, and the distinguishing characteristics of these systems can be described far more thoroughly than we have done.

6.29. An LTI system is said to have *phase lead* at a particular frequency $\omega = \omega_0$ if $\sphericalangle H(j\omega_0) > 0$. The terminology stems from the fact that if $e^{j\omega_0 t}$ is the input to this system, then the phase of the output will exceed, or lead, the phase of the input. Similarly, if $\sphericalangle H(j\omega_0) < 0$, the system is said to have *phase lag* at this frequency. Note that the system with frequency response

$$\frac{1}{1 + j\omega\tau}$$

has phase lag for all $\omega > 0$, while the system with frequency response

$$1 + j\omega\tau$$

has phase lead for all $\omega > 0$.

(a) Construct the Bode plots for the following two systems. Which has phase lead and which phase lag? Also, which one amplifies signals at certain frequencies?

(i) $\frac{1+(j\omega/10)}{1+10j\omega}$ **(ii)** $\frac{1+10j\omega}{1+(j\omega/10)}$

(b) Repeat part (a) for the following three frequency responses:

(i) $\frac{(1+(j\omega/10))^2}{(1+10j\omega)^3}$ **(ii)** $\frac{1+j\omega/10}{100(j\omega)^2+10j\omega+1}$ **(iii)** $\frac{1+10j\omega}{0.01(j\omega)^2+0.2j\omega+1}$

6.30. Let $h(t)$ have the Bode plot depicted in Figure P6.30. The dashed lines in the figure represent straight-line approximations. Sketch the Bode plots for $10h(10t)$.

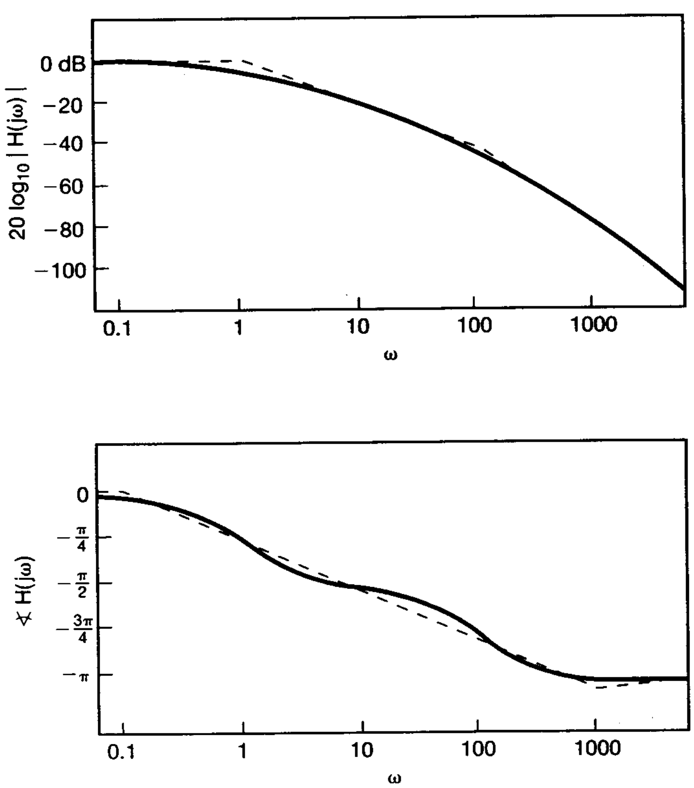


Figure P6.30

6.31. An integrator has as its frequency response

$$H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega),$$

where the impulse at $\omega = 0$ is a result of the fact that the integration of a constant input from $t = -\infty$ results in an infinite output. Thus, if we avoid inputs that are constant, or equivalently, only examine $H(j\omega)$ for $\omega > 0$, we see that

$$20 \log |H(j\omega)| = -20 \log(\omega),$$

$$\sphericalangle H(j\omega) = \frac{-\pi}{2}.$$

In other words, the Bode plot for an integrator, as illustrated in Figure P6.31, consists of two straight-line plots. These plots reflect the principal characteristics of an integrator: a phase shift of $-90°$ at all positive values of frequency and the amplification of low frequencies.

(a) A useful, simple model of an electric motor is an LTI system with input equal to the applied voltage and output given by the motor shaft angle. This system can be visualized as the cascade of a stable LTI system (with the voltage as input and shaft angular velocity as output) and an integrator (representing the integration of the angular velocity). Often, a model of first-order system is used for the first part of the cascade. Assuming, for example, that this first-order system has a time constant of 0.1 second, we obtain an overall motor frequency response of

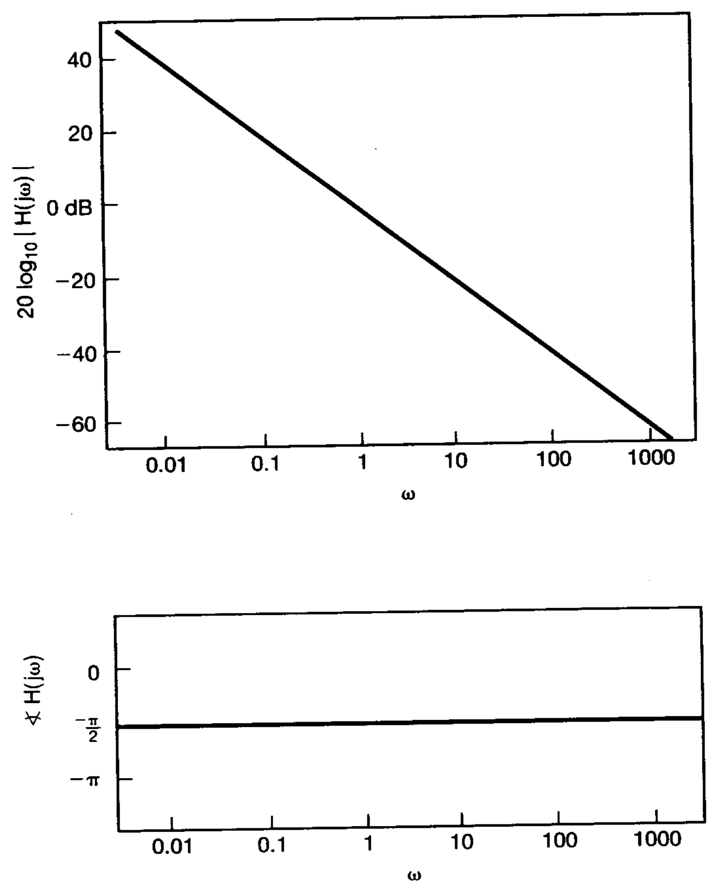


Figure P6.31

the form

$$H(j\omega) = \frac{1}{j\omega(1 + j\omega/10)} + \pi\,\delta(\omega).$$

Sketch the Bode plot for the system for $\omega > 0.001$.

(b) Sketch the Bode plot for a differentiator.

(c) Do the same for systems with the following frequency responses:

(i) $H(j\omega) = \frac{j\omega}{1+j\omega/100}$

(ii) $H(j\omega) = \frac{j\omega}{(1+(j\omega)/10+(j\omega)^2/100)}$

6.32. Consider the system depicted in Figure P6.32. This "compensator" box is a continuous-time LTI system.

(a) Suppose that it is desired to choose the frequency response of the compensator so that the overall frequency response $H(j\omega)$ of the cascade satisfies the following two conditions:

1. The log magnitude of $H(j\omega)$ has a slope of -40 dB/decade beyond $\omega = 1{,}000$.
2. For $0 < \omega < 1{,}000$, the log magnitude of $H(j\omega)$ should be between -10 dB and 10 dB.

Design a suitable compensator (that is, determine a frequency response for a compensator that meets the preceding requirements), and draw the Bode plot for the resulting $H(j\omega)$.

(b) Repeat (a) if the specifications on the log magnitude of $H(j\omega)$ are as follows:

1. It should have a slope of $+20$ dB/decade for $0 < \omega < 10$.
2. It should be between $+10$ and $+30$ dB for $10 < \omega < 100$.
3. It should have a slope of -20 dB/decade for $100 < \omega < 1{,}000$.
4. It should have a slope of -40 dB/decade for $\omega > 1{,}000$.

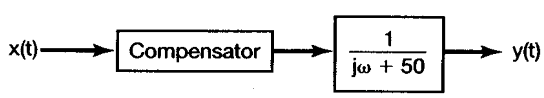


Figure P6.32

6.33. Figure P6.33 shows a system commonly used to obtain a highpass filter from a lowpass filter and vice versa.

(a) Show that, if $H(j\omega)$ is a lowpass filter with cutoff frequency ω_{lp}, the overall system corresponds to an ideal highpass filter. Determine the system's cutoff frequency and sketch its impulse response.

(b) Show that, if $H(j\omega)$ is an ideal highpass filter with cutoff frequency ω_{hp}, the overall system corresponds to an ideal lowpass filter, and determine the cutoff frequency of the system.

(c) If the interconnection of Figure P6.33 is applied to an ideal discrete-time lowpass filter, will the resulting system be an ideal discrete-time highpass filter?

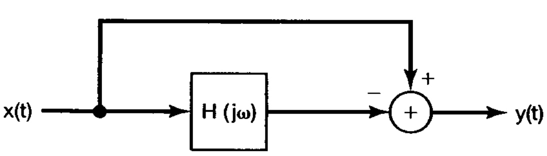


Figure P6.33

6.34. In Problem 6.33, we considered a system commonly used to obtain a highpass filter from a lowpass filter and vice versa. In this problem, we explore the system further and, in particular, consider a potential difficulty if the phase of $H(j\omega)$ is not properly chosen.

(a) Referring to Figure P6.33, let us assume that $H(j\omega)$ is real and as shown in Figure P6.34. Then

$$1 - \delta_1 < H(j\omega) < 1 + \delta_1, \quad 0 \le \omega \le \omega_1,$$

$$-\delta_2 < H(j\omega) < +\delta_2, \quad \omega_2 < \omega.$$

Determine and sketch the resulting frequency response of the overall system of Figure P6.33. Does the resulting system correspond to an approximation to a highpass filter?

(b) Now let $H(j\omega)$ in Figure P6.33 be of the form

$$H(j\omega) = H_1(j\omega)e^{j\theta(\omega)}, \qquad \text{(P6.34–1)}$$

where $H_1(j\omega)$ is identical to Figure P6.34 and $\theta(\omega)$ is an unspecified phase characteristic. With $H(j\omega)$ in this more general form, does it still correspond to an approximation to a lowpass filter?

(c) Without making any assumptions about $\theta(\omega)$, determine and sketch the tolerance limits on the magnitude of the frequency response of the overall system of Figure P6.33.

(d) If $H(j\omega)$ in Figure P6.33 is an approximation to a lowpass filter with unspecified phase characteristics, will the overall system in that figure necessarily correspond to an approximation to a highpass filter?

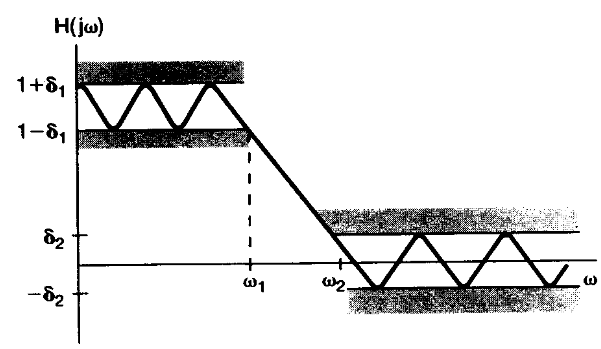


Figure P6.34

6.35. Shown in Figure P6.35 is the frequency response $H(e^{j\omega})$ of a discrete-time differentiator. Determine the output signal $y[n]$ as a function of ω_0 if the input $x[n]$ is

$$x[n] = \cos[\omega_0 n + \theta].$$

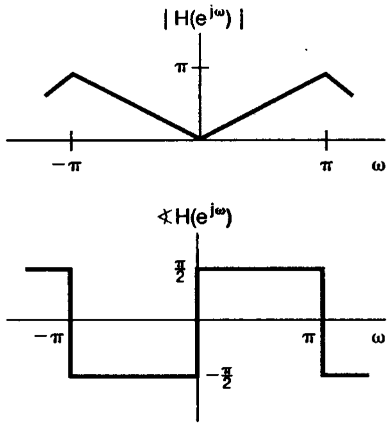


Figure P6.35

6.36. Consider a discrete-time lowpass filter whose impulse response $h[n]$ is known to be real and whose frequency response magnitude in the region $-\pi \leq \omega \leq \pi$ is given as:

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \leq \frac{\pi}{4} \\ 0, & \text{otherwise} \end{cases}.$$

Determine and sketch the real-valued impulse response $h[n]$ for this filter when the corresponding group delay function is specified as:
(a) $\tau(\omega) = 5$ **(b)** $\tau(\omega) = \frac{5}{2}$ **(c)** $\tau(\omega) = -\frac{5}{2}$

6.37. Consider a causal LTI system whose frequency response is given as:

$$H(e^{j\omega}) = e^{-j\omega}\frac{1 - \frac{1}{2}e^{j\omega}}{1 - \frac{1}{2}e^{-j\omega}}.$$

(a) Show that $|H(e^{j\omega})|$ is unity at all frequencies.
(b) Show that

$$\sphericalangle H(e^{j\omega}) = -\omega - 2\tan^{-1}\left(\frac{\frac{1}{2}\sin\omega}{1 - \frac{1}{2}\cos\omega}\right).$$

(c) Show that the group delay for this filter is given by

$$\tau(\omega) = \frac{\frac{3}{4}}{\frac{5}{4} - \cos\omega}.$$

Sketch $\tau(\omega)$.
(d) What is the output of this filter when the input is $\cos(\frac{\pi}{3}n)$?

6.38. Consider an ideal bandpass filter whose frequency response in the region $-\pi \leq \omega \leq \pi$ is specified as

$$H(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{2} - \omega_c \leq |\omega| \leq \frac{\pi}{2} + \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

Determine and sketch the impulse response $h[n]$ for this filter when

(a) $\omega_c = \frac{\pi}{5}$

(b) $\omega_c = \frac{\pi}{4}$

(c) $\omega_c = \frac{\pi}{3}$

As ω_c is increased, does $h[n]$ get more or less concentrated about the origin?

6.39. Sketch the log magnitude and phase of each of the following frequency responses.

(a) $1 + \frac{1}{2}e^{-j\omega}$ **(b)** $1 + 2e^{-j\omega}$

(c) $1 - 2e^{-j\omega}$ **(d)** $1 + 2e^{-j2\omega}$

(e) $\frac{1}{(1+\frac{1}{2}e^{-j\omega})^3}$ **(f)** $\frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}$

(g) $\frac{1+2e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$ **(h)** $\frac{1-2e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$

(i) $\frac{1}{(1-\frac{1}{4}e^{-j\omega})(1-\frac{3}{4}e^{-j\omega})}$ **(j)** $\frac{1}{(1-\frac{1}{4}e^{-j\omega})(1+\frac{3}{4}e^{-j\omega})}$

(k) $\frac{1+2e^{-2j\omega}}{(1-\frac{1}{2}e^{-j\omega})^2}$

6.40. Consider an ideal discrete-time lowpass filter with impulse response $h[n]$ and for which the frequency response $H(e^{j\omega})$ is that shown in Figure P6.40. Let us consider obtaining a new filter with impulse response $h_1[n]$ and frequency response $H_1(e^{j\omega})$ as follows:

$$h_1[n] = \begin{cases} h[n/2], & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

This corresponds to inserting a sequence value of zero between each sequence value of $h[n]$. Determine and sketch $H_1(e^{j\omega})$ and state the class of ideal filters to which it belongs (e.g., lowpass, highpass, bandpass, multiband, etc.).

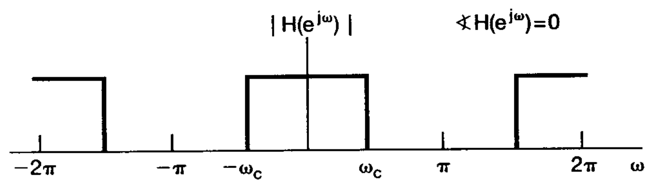


Figure P6.40

6.41. A particular causal LTI system is described by the difference equation

$$y[n] - \frac{\sqrt{2}}{2}y[n-1] + \frac{1}{4}y[n-2] = x[n] - x[n-1].$$

(a) Find the impulse response of this system.

(b) Sketch the log magnitude and the phase of the frequency response of the system.

6.42. **(a)** Consider two LTI systems with the following frequency responses:

$$H_1(e^{j\omega}) = \frac{1 + \frac{1}{2}e^{-j\omega}}{1 + \frac{1}{4}e^{-j\omega}},$$

$$H_2(e^{j\omega}) = \frac{\frac{1}{2} + e^{-j\omega}}{1 + \frac{1}{4}e^{-j\omega}}.$$

Show that both of these frequency responses have the same magnitude function [i.e., $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$], but the group delay of $H_2(e^{j\omega})$ is greater than the group delay of $H_1(e^{j\omega})$ for $\omega > 0$.

(b) Determine and sketch the impulse and step responses of the two systems.

(c) Show that

$$H_2(e^{j\omega}) = G(e^{j\omega})H_1(e^{j\omega}),$$

where $G(e^{j\omega})$ is an *all-pass system* [i.e., $|G(e^{j\omega})| = 1$ for all ω].

6.43. When designing filters with highpass or bandpass characteristics, it is often convenient first to design a lowpass filter with the desired passband and stopband specifications and then to transform this prototype filter to the desired highpass or bandpass filter. Such transformations are called lowpass-to-highpass or highpass-to-lowpass transformations. Designing filters in this manner is convenient because it requires us only to formulate our filter design algorithms for the class of filters with lowpass characteristics. As one example of such a procedure, consider a discrete-time lowpass filter with impulse response $h_{lp}[n]$ and frequency response $H_{lp}(e^{j\omega})$, as sketched in Figure P6.43. Suppose the impulse response is modulated with the sequence $(-1)^n$ to obtain $h_{hp}[n] = (-1)^n h_{lp}[n]$.

(a) Determine and sketch $H_{hp}(e^{j\omega})$ in terms of $H_{lp}(e^{j\omega})$. Show in particular that, for $H_{lp}(e^{j\omega})$ as shown in Figure P6.43, $H_{hp}(e^{j\omega})$ corresponds to a highpass filter.

(b) Show that modulation of the impulse response of a discrete-time highpass filter by $(-1)^n$ will transform it to a lowpass filter.

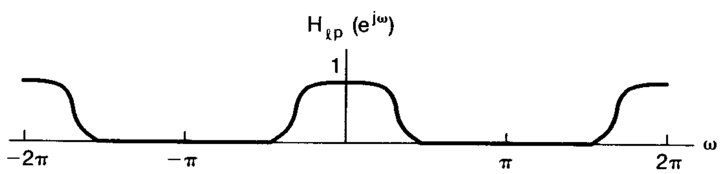


Figure P6.43

6.44. A discrete-time system is implemented as shown in Figure P6.44. The system S shown in the figure is an LTI system with impulse response $h_{lp}[n]$.

(a) Show that the overall system is time invariant.

(b) If $h_{lp}[n]$ is a lowpass filter, what type of filter does the system of the figure implement?

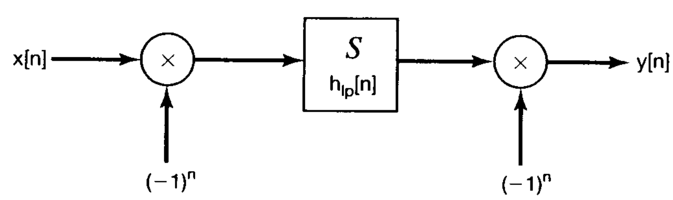


Figure P6.44

6.45. Consider the following three frequency responses for causal and stable third-order LTI systems. By utilizing the properties of first- and second-order systems discussed in Section 6.6, determine whether or not the impulse response of each of the third-order systems is oscillatory. (*Note:* You should be able to answer this question without taking the inverse Fourier transforms of the frequency responses of the third-order systems.)

$$H_1(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})},$$

$$H_2(e^{j\omega}) = \frac{1}{(1+\frac{1}{2}e^{-j\omega})(1-\frac{1}{3}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})},$$

$$H_3(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{3}{4}e^{-j\omega}+\frac{9}{16}e^{-j2\omega})}.$$

6.46. Consider a causal, nonrecursive (FIR) filter whose real-valued impulse response $h[n]$ is zero for $n \geq N$.

(a) Assuming that N is odd, show that if $h[n]$ is symmetric about $(N-1)/2$ (i.e., if $h[(N-1)/2+n] = h[(N-1)/2-n]$), then

$$H(e^{j\omega}) = A(\omega)e^{-j[(N-1)/2]\omega},$$

where $A(\omega)$ is a real-valued function of ω. We conclude that the filter has linear phase.

(b) Give an example of the impulse response $h[n]$ of a causal, linear-phase FIR filter such that $h[n] = 0$ for $n \geq 5$ and $h[n] \neq 0$ for $0 \leq n \leq 4$.

(c) Assuming that N is even, show that if $h[n]$ is symmetric about $(N-1)/2$ (i.e., if $h[(N/2)+n] = h[N/2-n-1]$), then

$$H(e^{j\omega}) = A(\omega)e^{-j[(N-1)/2]\omega},$$

where $A(\omega)$ is a real-valued function of ω.

(d) Give an example of the impulse response $h[n]$ of a causal, linear-phase FIR filter such that $h[n] = 0$ for $n \geq 4$ and $h[n] \neq 0$ for $0 \leq n \leq 3$.

6.47. A three-point symmetric moving average, referred to as a weighted moving average, is of the form

$$y[n] = b\{ax[n-1] + x[n] + ax[n+1]\}. \qquad \text{(P6.47–1)}$$

(a) Determine, as a function of a and b, the frequency response $H(e^{j\omega})$ of the three-point moving average in eq. (P6.47–1).

(b) Determine the scaling factor b such that $H(e^{j\omega})$ has unity gain at zero frequency.

(c) In many time-series analysis problems, a common choice for the coefficient a in the weighted moving average in eq. (P6.47–1) is $a = 1/2$. Determine and sketch the frequency response of the resulting filter.

6.48. Consider a four-point, moving-average, discrete-time filter for which the difference equation is

$$y[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + b_3x[n-2].$$

Determine and sketch the magnitude of the frequency response for each of the following cases:

(a) $b_0 = b_3 = 0, b_1 = b_2$
(b) $b_1 = b_2 = 0, b_0 = b_3$
(c) $b_0 = b_1 = b_2 = b_3$
(d) $b_0 = -b_1 = b_2 = -b_3$

ADVANCED PROBLEMS

6.49. The time constant provides a measure of how fast a first-order system responds to inputs. The idea of measuring the speed of response of a system is also important for higher order systems, and in this problem we investigate the extension of the time constant to such systems.

(a) Recall that the time constant of a first-order system with impulse response

$$h(t) = ae^{-at}u(t), \quad a > 0,$$

is $1/a$, which is the amount of time from $t = 0$ that it takes the system step response $s(t)$ to settle within $1/e$ of its final value [i.e., $s(\infty) = \lim_{t\to\infty} s(t)$]. Using this same quantitative definition, find the equation that must be solved in order to determine the time constant of the causal LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 11\frac{dy(t)}{dt} + 10y(t) = 9x(t). \qquad \text{(P6.49–1)}$$

(b) As can be seen from part (a), if we use the precise definition of the time constant set forth there, we obtain a simple expression for the time constant of a first-order system, but the calculations are decidedly more complex for the system of eq. (P6.49–1). However, show that this system can be viewed as the parallel interconnection of two first-order systems. Thus, we usually think of the system of eq. (P6.49–1) as having *two* time constants, corresponding to the two first-order factors. What are the two time constants for this system?

(c) The discussion given in part (b) can be directly generalized to all systems with impulse responses that are linear combinations of decaying exponentials. In any system of this type, one can identify the *dominant* time constants of the system, which are simply the largest of the time constants. These represent the slowest parts of the system response, and consequently, they have the dominant effect on how fast the system as a whole can respond. What is the dominant time constant of the system of eq. (P6.49–1)? Substitute this time constant into the equation determined in part (a). Although the number will not satisfy the equation exactly, you should see that it nearly does, which is an indication that it is very close to the time constant defined in part (a). Thus, the approach we have outlined in part (b) and here is of value in providing insight into the speed of response of LTI systems without requiring excessive calculation.

(d) One important use of the concept of dominant time constants is in the reduction of the order of LTI systems. This is of great practical significance in problems

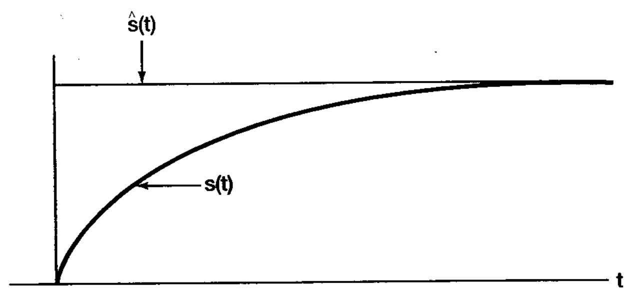


Figure P6.49

involving the analysis of complex systems having a few dominant time constants and other very small time constants. In order to reduce the complexity of the model of the system to be analyzed, one often can simplify the fast parts of the system. That is, suppose we regard a complex system as a parallel interconnection of first- and second-order systems. Suppose also that one of these subsystems, with impulse response $h(t)$ and step response $s(t)$, is fast—that is, that $s(t)$ settles to its final value $s(\infty)$ very quickly. Then we can approximate this subsystem by the subsystem that settles to the same final value *instantaneously*. That is, if $\hat{s}(t)$ is the step response to our approximation, then

$$\hat{s}(t) = s(\infty)u(t).$$

This is illustrated in Figure P6.49. Note that the impulse response of the approximate system is then

$$\hat{h}(t) = s(\infty)\delta(t),$$

which indicates that the approximate system is *memoryless*.

Consider again the causal LTI system described by eq. (P6.49–1) and, in particular, the representation of it as a parallel interconnection of two first-order systems, as described in part (b). Use the method just outlined to replace the faster of the two subsystems by a memoryless system. What is the differential equation that then describes the resulting overall system? What is the frequency response of this system? Sketch $|H(j\omega)|$ (not $\log|H(j\omega)|$) and $\sphericalangle H(j\omega)$ for both the original and approximate systems. Over what range of frequencies are these frequency responses nearly equal? Sketch the step responses for both systems. Over what range of time are the step responses nearly equal? From your plots, you will see some of the similarities *and* differences between the original system and its approximation. The utility of an approximation such as this depends upon the specific application. In particular, one must take into account both how widely separated the different time constants are and also the nature of the inputs to be considered. As you will see from your answers in this part of the problem, the frequency response of the approximate system is essentially the same as the frequency response of the original system at low frequencies. That is, when the fast parts of the system are sufficiently fast compared to the rate of fluctuation of the input, the approximation becomes useful.

6.50. The concepts associated with frequency-selective filtering are often used to separate two signals that have been added together. If the spectra of the two signals do not overlap, ideal frequency-selective filters are desirable. However, if the spectra overlap, it is often preferable to design the filter to have a gradual transition between passband and stopband. In this problem, we explore one approach for determining the frequency response of a filter to be used for separating signals with overlapping spectra. Let $x(t)$ denote a composite continuous-time signal consisting of the sum of two signals $s(t) + w(t)$. As indicated in Figure P6.50(a), we would like to design an LTI filter to recover $s(t)$ from $x(t)$. The filter's frequency response $H(j\omega)$ is to be chosen so that, in some sense, $y(t)$ is a "good" approximation to $s(t)$.

Let us define a measure of the error between $y(t)$ and $s(t)$ at each frequency ω as

$$\epsilon(\omega) \triangleq |S(j\omega) - Y(j\omega)|^2,$$

where $S(j\omega)$ and $Y(j\omega)$ are the Fourier transforms of $s(t)$ and $y(t)$, respectively.

(a) Express $\epsilon(\omega)$ in terms of $S(j\omega)$, $H(j\omega)$, and $W(j\omega)$, where $W(j\omega)$ is the Fourier transform of $w(t)$.

(b) Let us restrict $H(j\omega)$ to be real, so that $H(j\omega) = H^*(j\omega)$. By setting the derivative of $\epsilon(\omega)$ with respect to $H(j\omega)$ to be zero, determine the $H(j\omega)$ required to minimize the error $\epsilon(\omega)$.

(c) Show that if the spectra of $S(j\omega)$ and $W(j\omega)$ are non-overlapping, the result in part (b) reduces to an ideal frequency-selective filter.

(d) From your result in part (b), determine and sketch $H(j\omega)$ if $S(j\omega)$ and $W(j\omega)$ are as shown in Figure P6.50(b).

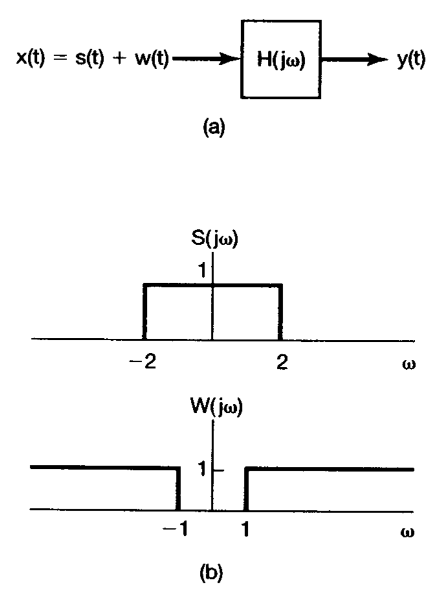


Figure P6.50

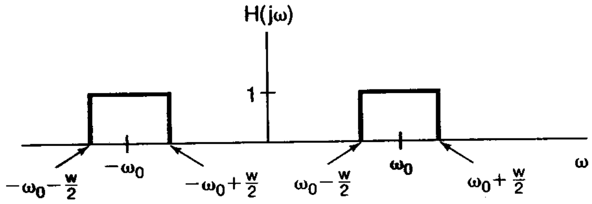
6.51. An ideal bandpass filter is a bandpass filter that passes only a range of frequencies, without any change in amplitude or phase. As shown in Figure P6.51(a), let the passband be

$$\omega_0 - \frac{W}{2} \le |\omega| \le \omega_0 + \frac{W}{2}.$$

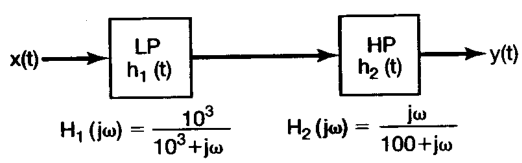
(a) What is the impulse response $h(t)$ of this filter?

(b) We can approximate an ideal bandpass filter by cascading a first-order lowpass and a first-order highpass filter, as shown in Figure P6.51(b). Sketch the Bode diagrams for each of the two filters $H_1(j\omega)$ and $H_2(j\omega)$.

(c) Determine the Bode diagram for the overall bandpass filter in terms of your results from part (b).



(a)



(b)

Figure P6.51

6.52. In Figure P6.52(a), we show the magnitude of the frequency response for an ideal continuous-time differentiator. A nonideal differentiator would have a frequency response that is some approximation to the frequency response in the figure.

(a) Consider a nonideal differentiator with frequency response $G(j\omega)$ for which $|G(j\omega)|$ is constrained to be within $\pm 10\%$ of the magnitude of the frequency response of the ideal differentiator at all frequencies; that is,

$$-0.1|H(j\omega)| \le [|G(j\omega)| - |H(j\omega)|] \le 0.1|H(j\omega)|.$$

Sketch the region in a plot of $G(j\omega)$ vs. ω where $|G(j\omega)|$ must be confined to meet this specification.

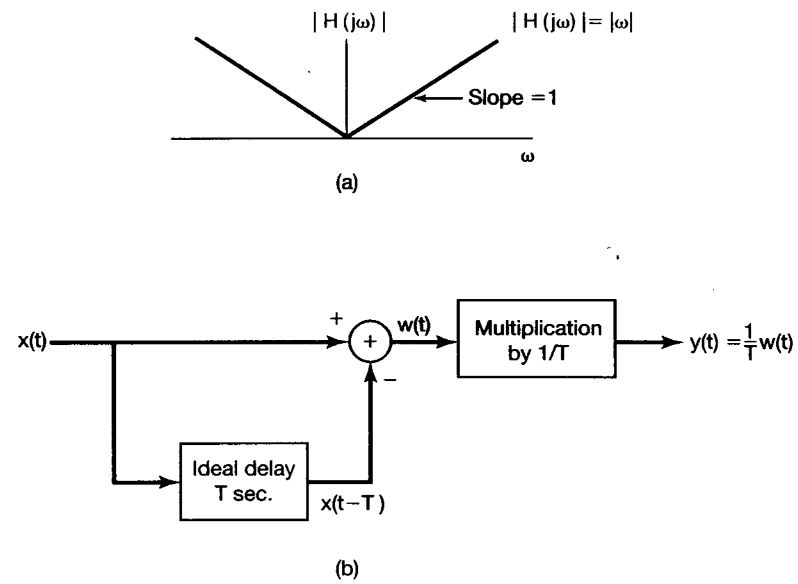


Figure P6.52

(b) The system in Figure P6.52(b), incorporating an ideal delay of T seconds, is sometimes used to approximate a continuous-time differentiator. For $T = 10^{-2}$ second, determine the frequency range over which the magnitude of the frequency response of the system in the figure is within $\pm 10\%$ of that for an ideal differentiator.

6.53. In many filtering applications, it is often undesirable for the step response of a filter to overshoot its final value. In processing pictures, for example, the overshoot in the step response of a linear filter may produce flare—that is, an increase in intensity—at sharp boundaries. It is possible, however, to eliminate overshoot by requiring that the impulse response of the filter be positive for all time.

Show that if $h(t)$, the impulse response of a continuous-time LTI filter, is always greater than or equal to zero, the step response of the filter is a monotonically nondecreasing function and therefore will not have overshoot.

6.54. By means of a specific filter design procedure, a nonideal continuous-time lowpass filter with frequency response $H_0(j\omega)$, impulse response $h_0(t)$, and step response $s_0(t)$ has been designed. The cutoff frequency of the filter is at $\omega = 2\pi \times 10^2$ rad/sec, and the step response rise time, defined as the time required for the step response to go from 10% of its final value to 90% of its final value, is $\tau_r = 10^{-2}$ second. From this design, we can obtain a new filter with an arbitrary cutoff frequency ω_c by the use of frequency scaling. The frequency response of the resulting filter is then of the form

$$H_{\text{lp}}(j\omega) = H_0(ja\omega),$$

where a is an appropriate scale factor.

(a) Determine the scale factor a such that $H_{\text{lp}}(j\omega)$ has a cutoff frequency of ω_c.

(b) Determine the impulse response $h_{\text{lp}}(t)$ of the new filter in terms of ω_c and $h_0(t)$.
(c) Determine the step response $s_{\text{lp}}(t)$ of the new filter in terms of ω_c and $s_0(t)$.
(d) Determine and sketch the rise time of the new filter as a function of its cutoff frequency ω_c.

This is one illustration of the trade-off between time-domain and frequency-domain characteristics. In particular, as the cutoff frequency decreases, the rise time tends to increase.

6.55. The square of the magnitude of the frequency response of a class of continuous-time lowpass filters, known as Butterworth filters, is

$$|B(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2N}}.$$

Let us define the passband edge frequency ω_p as the frequency below which $|B(j\omega)|^2$ is greater than one-half of its value at $\omega = 0$; that is,

$$|B(j\omega)|^2 \geq \frac{1}{2}|B(j0)|^2, \quad |\omega| < \omega_p.$$

Now let us define the stopband edge frequency ω_s as the frequency above which $|B(j\omega)|^2$ is less than 10^{-2} of its value at $\omega = 0$; that is,

$$|B(j\omega)|^2 \leq 10^{-2}|B(j0)|^2, \quad |\omega| > \omega_s.$$

The transition band is then the frequency range between ω_p and ω_s. The ratio ω_s/ω_p is referred to as the transition ratio.

For fixed ω_p, and making reasonable approximations, determine and sketch the transition ratio as a function of N for the class of Butterworth filters.

6.56. In this problem, we explore some of the filtering issues involved in the commercial version of a typical system that is used in most modern cassette tape decks to reduce noise. The primary source of noise is the high-frequency hiss in the tape playback process, which, in some part, is due to the friction between the tape and the playback head. Let us assume that the noise hiss that is added to the signal upon playback has the spectrum of Figure P6.56(a) when measured in decibels, with 0 dB equal to the signal level at 100 Hz. The spectrum $S(j\omega)$ of the signal has the shape shown in Figure P6.56(b).

The system that we analyze has a filter $H_1(j\omega)$ which conditions the signal $s(t)$ before it is recorded. Upon playback, the hiss $n(t)$ is added to the signal. The system is represented schematically in Figure P6.56(c).

Suppose we would like our overall system to have a signal-to-noise ratio of 40 dB over the frequency range 50 Hz $< \omega/2\pi <$ 20 kHz.

(a) Determine the transfer characteristic of the filter $H_1(j\omega)$. Sketch the Bode plot of $H_1(j\omega)$.
(b) If we were to listen to the signal $p(t)$, assuming that the playback process does nothing more than add hiss to the signal, how do you think it would sound?
(c) What should the Bode plot and transfer characteristic of the filter $H_2(j\omega)$ be in order for the signal $\hat{s}(t)$ to sound similar to $s(t)$?

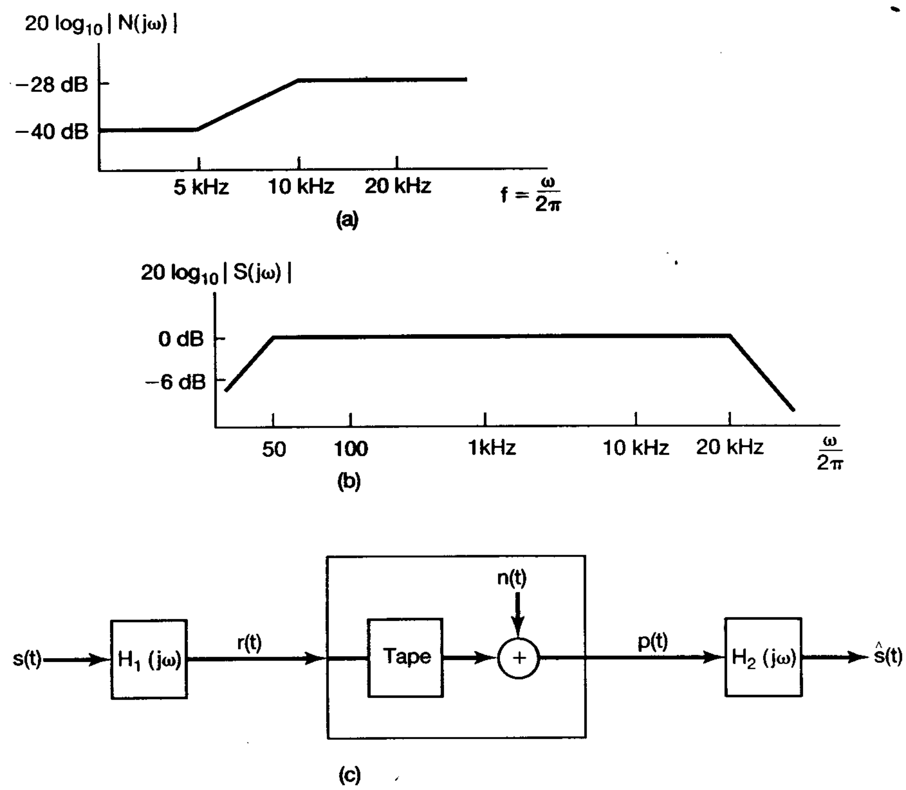


Figure P6.56

6.57. Show that if $h[n]$, the impulse response of a discrete-time LTI filter, is always greater than or equal to zero, the step response of the filter is a monotonically nondecreasing function and therefore will not have overshoot.

6.58. In the design of either analog or digital filters, we often approximate a specified magnitude characteristic without particular regard to the phase. For example, standard design techniques for lowpass and bandpass filters are typically derived from a consideration of the magnitude characteristics only.

In many filtering problems, one would ideally like the phase characteristics to be zero or linear. For causal filters, it is impossible to have zero phase. However, for many digital filtering applications, it is not necessary that the unit sample response of the filter be zero for $n < 0$ if the processing is not to be carried out in real time.

One technique commonly used in digital filtering when the data to be filtered are of finite duration and stored, for example, on a disc or magnetic tape is to process the data forward and then backward through the same filter.

Let $h[n]$ be the unit sample response of a causal filter with an arbitrary phase characteristic. Assume that $h[n]$ is real, and denote its Fourier transform by $H(e^{j\omega})$. Let $x[n]$ be the data that we want to filter. The filtering operation is performed as follows:

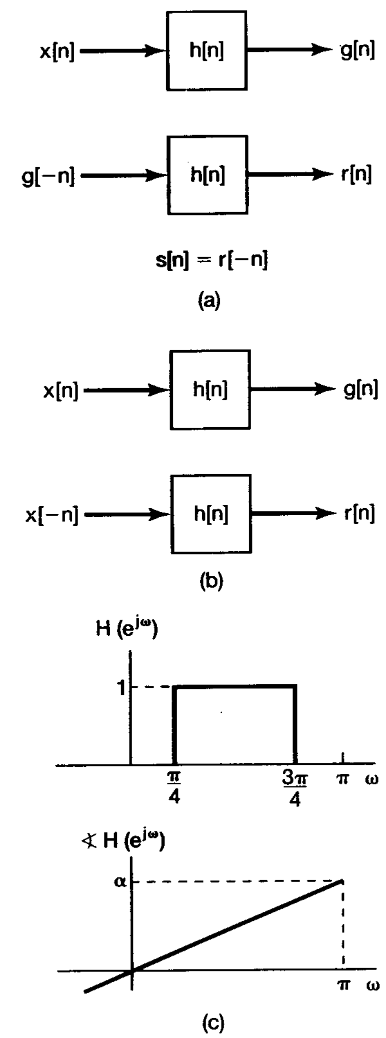


Figure P6.58

(a) *Method A:* Process $x[n]$ to get $s[n]$, as indicated in Figure P6.58(a).

1. Determine the overall unit sample response $h_1[n]$ that relates $x[n]$ and $s[n]$, and show that it has zero phase characteristic.

2. Determine $|H_1(e^{j\omega})|$ and express it in terms of $|H(e^{j\omega})|$ and $\sphericalangle H(e^{j\omega})$.

(b) *Method B:* Process $x[n]$ through the filter $h[n]$ to get $g[n]$ [Figure P6.58(b)]. Also, process $x[n]$ backward through $h[n]$ to get $r[n]$. The output $y[n]$ is taken to be the sum of $g[n]$ and $r[-n]$. The composite set of operations can be represented by a filter with input $x[n]$, output $y[n]$, and unit sample response $h_2[n]$.

1. Show that the composite filter $h_2[n]$ has zero phase characteristic.

2. Determine $|H_2(e^{j\omega})|$, and express it in terms of $|H(e^{j\omega})|$ and $\sphericalangle H(e^{j\omega})$.

(c) Suppose that we are given a sequence of finite duration on which we would like to perform bandpass, zero-phase filtering. Furthermore, assume that we are given the bandpass filter $h[n]$ with frequency response as specified in Figure P6.58(c) and with magnitude characteristic that we desire, but with linear phase. To achieve zero phase, we could use either of the preceding methods, A or B. Determine and sketch $|H_1(e^{j\omega})|$ and $|H_2(e^{j\omega})|$. From these results, which method would you use to achieve the desired bandpass filtering operation? Explain why. More generally, if $h[n]$ has the desired magnitude, but a nonlinear phase characteristic, which method is preferable to achieve a zero phase characteristic?

6.59. Let $h_d[n]$ denote the unit sample response of a desired ideal system with frequency response $H_d(e^{j\omega})$, and let $h[n]$ denote the unit sample response for an FIR system of length N and with frequency response $H(e^{j\omega})$. In this problem, we show that a rectangular window of length N samples applied to $h_d[n]$ will produce a unit sample response $h[n]$ such that the mean square error

$$\epsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_d(e^{j\omega}) - H(e^{j\omega})|^2\, d\omega$$

is minimized.

(a) The error function $E(e^{j\omega}) = H_d(e^{j\omega}) - H(e^{j\omega})$ can be expressed as the power series

$$E(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e[n]e^{-j\omega n}.$$

Find the coefficients $e[n]$ in terms of $h_d[n]$ and $h[n]$.

(b) Using Parseval's relation, express the mean square error ϵ^2 in terms of the coefficients $e[n]$.

(c) Show that for a unit sample response $h[n]$ of length N samples, ϵ^2 is minimized when

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}.$$

That is, simple truncation gives the best mean square approximation to a desired frequency response for a fixed value of N.

6.60. In Problem 6.50, we considered one specific criterion for determining the frequency response of a continuous-time filter that would recover a signal from the sum of two signals when their spectra overlapped in frequency. For the discrete-time case, develop the result corresponding to that obtained in part (b) of Problem 6.50.

6.61. In many situations we have available an analog or digital filter module, such as a basic hardware element or computer subroutine. By using the module repetitively or by combining identical modules, it is possible to implement a new filter with improved passband or stopband characteristics. In this and the next problem, we consider two procedures for doing just that. Although the discussion is phrased in terms of discrete-time filters, much of it applies directly to continuous-time filters as well.

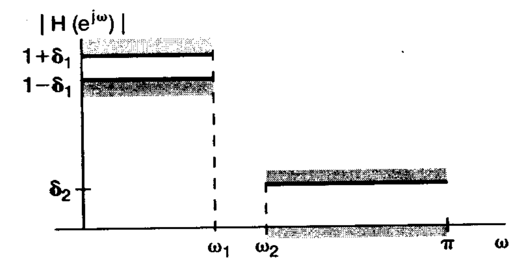


Figure P6.61

Consider a lowpass filter with frequency response $H(e^{j\omega})$ for which $|H(e^{j\omega})|$ falls within the tolerance limits shown in Figure P6.61; that is,

$$1 - \delta_1 \leq |H(e^{j\omega})| \leq 1 + \delta_1, \quad 0 \leq \omega \leq \omega_1,$$
$$0 \leq |H(e^{j\omega})| \leq \delta_2, \quad \omega_2 \leq \omega \leq \pi.$$

A new filter with frequency response $G(e^{j\omega})$ is formed by cascading two identical filters, both with frequency response $H(e^{j\omega})$.

(a) Determine the tolerance limits on $|G(e^{j\omega})|$.

(b) Assuming that $H(e^{j\omega})$ is a good approximation to a lowpass filter, so that $\delta_1 << 1$ and $\delta_2 << 1$, determine whether the passband ripple for $G(e^{j\omega})$ is larger or smaller than the passband ripple for $H(e^{j\omega})$. Also, determine whether the stopband ripple for $G(e^{j\omega})$ is larger or smaller than the stopband ripple for $H(e^{j\omega})$.

(c) If N identical filters with frequency response $H(e^{j\omega})$ are cascaded to obtain a new frequency response $G(e^{j\omega})$, then, again assuming that $\delta_1 << 1$ and $\delta_2 << 1$, determine the approximate tolerance limits on $|G(e^{j\omega})|$.

6.62. In Problem 6.61, we considered one method for using a basic filter module repetitively to implement a new filter with improved characteristics. Let us now consider an alternative approach, proposed by J. W. Tukey in the book, *Exploratory Data Analysis* (Reading, MA: Addison-Wesley Publishing Co., Inc., 1976). The procedure is shown in block diagram form in Figure P6.62(a).

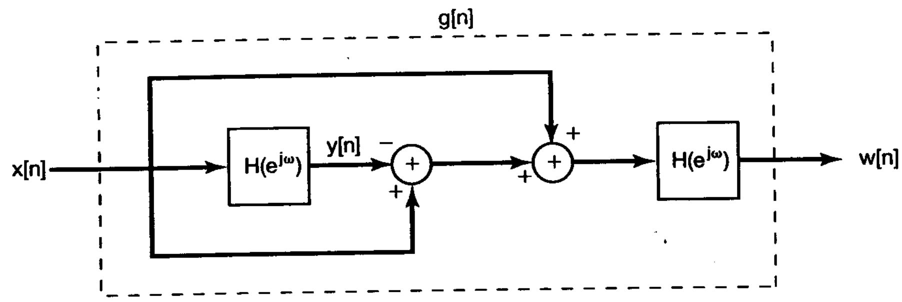
(a) Suppose that $H(e^{j\omega})$ is real and has a passband ripple of $\pm\delta_1$ and a stopband ripple of $\pm\delta_2$ (i.e., $H(e^{j\omega})$ falls within the tolerance limits indicated in Figure P6.62(b)). The frequency response $G(e^{j\omega})$ of the overall system in Figure P6.62(a) falls within the tolerance limits indicated in Figure P6.62(c). Determine A, B, C, and D in terms of δ_1 and δ_2.

(b) If $\delta_1 << 1$ and $\delta_2 << 1$, what is the approximate passband ripple and stopband ripple associated with $G(e^{j\omega})$? Indicate in particular whether the passband ripple for $G(e^{j\omega})$ is larger or smaller than the passband ripple for $H(e^{j\omega})$. Also, indicate whether the stopband ripple for $G(e^{j\omega})$ is larger or smaller than the stopband ripple for $H(e^{j\omega})$.

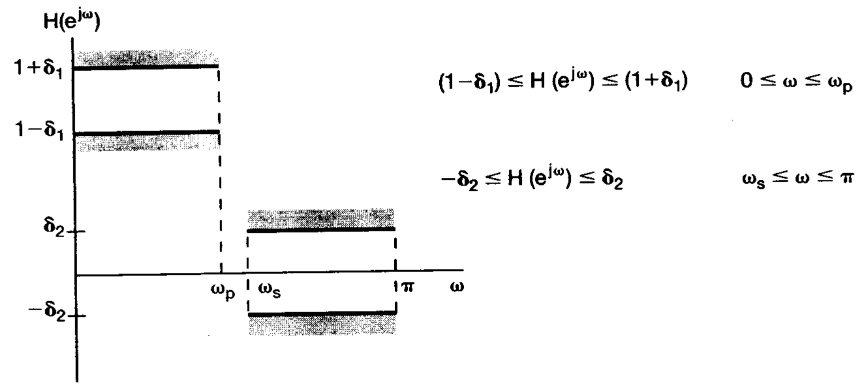
(c) In parts (a) and (b), we assumed that $H(e^{j\omega})$ is real. Now consider $H(e^{j\omega})$ to have the more general form

$$H(e^{j\omega}) = H_1(e^{j\omega})e^{j\theta(\omega)},$$

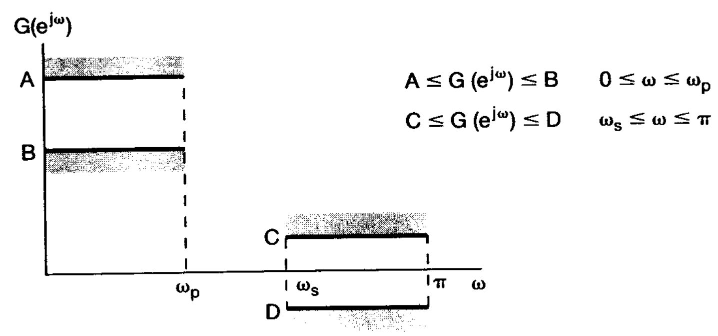
where $H_1(e^{j\omega})$ is real and $\theta(\omega)$ is an unspecified phase characteristic. If $|H(e^{j\omega})|$



(a)



(b)



(c)

Figure P6.62

is a reasonable approximation to an ideal lowpass filter, will $|G(e^{j\omega})|$ necessarily be a reasonable approximation to an ideal lowpass filter?

(d) Now assume that $H(e^{j\omega})$ is an FIR linear-phase lowpass filter, so that

$$H(e^{j\omega}) = H_1(e^{j\omega})e^{jM\omega},$$

where $H_1(e^{j\omega})$ is real and M is an integer. Show how to modify the system in Figure P6.62(a) so that the overall system will approximate a lowpass filter.

6.63. In the design of digital filters, we often choose a filter with a specified magnitude characteristic that has the shortest duration. That is, the impulse response, which is the inverse Fourier transform of the complex frequency spectrum, should be as narrow as possible. Assuming that $h[n]$ is real, we wish to show that if the phase $\theta(\omega)$ associated with the frequency response $H(e^{j\omega})$ is zero, the duration of the impulse response is minimal. Let the frequency response be expressed as

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)},$$

and let us consider the quantity

$$D = \sum_{n=-\infty}^{\infty} n^2 h^2[n] = \sum_{n=-\infty}^{\infty} (nh[n])^2$$

to be a measure of the duration of the associated impulse response $h[n]$.

(a) Using the derivative property of the Fourier transform and Parseval's relation, express D in terms of $H(e^{j\omega})$.

(b) By expressing $H(e^{j\omega})$ in terms of its magnitude $|H(e^{j\omega})|$ and phase $\theta(\omega)$, use your result from part (a) to show that D is minimized when $\theta(\omega) = 0$.

6.64. For a discrete-time filter to be *causal* and to have exactly linear phase, its impulse response must be of finite length and consequently the difference equation must be nonrecursive. To focus on the insight behind this statement, we consider a particular case, that of a linear phase characteristic for which the slope of the phase is an integer. Thus, the frequency response is assumed to be of the form

$$H(e^{j\omega}) = H_r(e^{j\omega})e^{-jM\omega}, \ -\pi < \omega < \pi \qquad \text{(P6.64–1)}$$

where $H_r(e^{j\omega})$ is real and even.

Let $h[n]$ denote the impulse response of the filter with frequency response $H(e^{j\omega})$ and let $h_r[n]$ denote the impulse response of the filter with frequency response $H_r(e^{j\omega})$.

(a) By using the appropriate properties in Table 5.1, show that:

1. $h_r[n] = h_r[-n]$ (i.e., $h_r[n]$ is symmetric about $n = 0$).
2. $h[n] = h_r[n - M]$.

(b) Using your result in part (a), show that with $H(e^{j\omega})$ of the form shown in eq. (P6.64–1), $h[n]$ is symmetric about $n = M$, that is,

$$h[M + n] = h[M - n]. \qquad \text{(P6.64–2)}$$

(c) According to the result in part (b), the linear phase characteristic in eq. (P6.64–1) imposes a symmetry in the impulse response. Show that if $h[n]$ is causal and has the symmetry in eq. (P6.64–2), then

$$h[n] = 0, \qquad n < 0 \text{ and } n > 2M$$

(i.e., it must be of finite length).

6.65. For a class of discrete-time lowpass filters, known as Butterworth filters, the squared magnitude of the frequency response is given by

$$|B(e^{j\omega})|^2 = \frac{1}{1 + \left(\frac{\tan(\omega/2)}{\tan(\omega_c/2)}\right)^{2N}},$$

where ω_c is the cutoff frequency (which we shall take to be $\pi/2$) and N is the order of the filter (which we shall consider to be $N = 1$). Thus, we have

$$|B(e^{j\omega})|^2 = \frac{1}{1 + \tan^2(\omega/2)}.$$

(a) Using trigonometric identities, show that $|B(e^{j\omega})|^2 = \cos^2(\omega/2)$.

(b) Let $B(e^{j\omega}) = a\cos(\omega/2)$. For what complex values of a is $|B(e^{j\omega})|^2$ the same as in part (a)?

(c) Show that $B(e^{j\omega})$ from part (b) is the transfer function corresponding to a difference equation of the form

$$y[n] = \alpha x[n] + \beta x[n - \gamma].$$

Determine α, β, and γ.

6.66. In Figure P6.66(a) we show a discrete-time system consisting of a parallel combination of N LTI filters with impulse response $h_k[n]$, $k = 0, 1, \cdots, N-1$. For any k, $h_k[n]$ is related to $h_0[n]$ by the expression

$$h_k[n] = e^{j(2\pi nk/N)} h_0[n].$$

(a) If $h_0[n]$ is an ideal discrete-time lowpass filter with frequency response $H_0(e^{j\omega})$ as shown in Figure P6.66(b), sketch the Fourier transforms of $h_1[n]$ and $h_{N-1}[n]$ for ω in the range $-\pi < \omega \leq +\pi$.

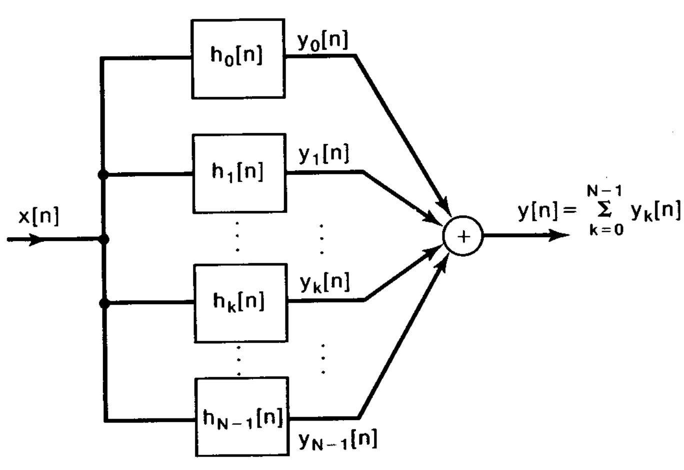


Figure P6.66a

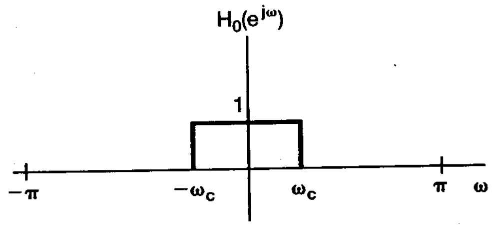


Figure P6.66b

(b) Determine the value of the cutoff frequency ω_c in Figure P6.66(b) in terms of N $(0 < \omega_c \leq \pi)$ such that the system of Figure P6.66(a) is an identity system; that is, $y[n] = x[n]$ for all n and any input $x[n]$.

(c) Suppose that $h[n]$ is no longer restricted to be an ideal lowpass filter. If $h[n]$ denotes the impulse response of the entire system in Figure P6.66(a) with input $x[n]$ and output $y[n]$, then $h[n]$ can be expressed in the form

$$h[n] = r[n]h_0[n].$$

Determine and sketch $r[n]$.

(d) From your result of part (c), determine a necessary and sufficient condition on $h_0[n]$ to ensure that the overall system will be an identity system (i.e., such that for any input $x[n]$, the output $y[n]$ will be identical to $x[n]$). Your answer should not contain any sums.