A Stable Marriages Algorithm to Optimize Satisfaction and Equity

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Abstract. This paper deals with designing an algorithm for feature pairing in vision, based on the "stable marriages" paradigm. Our *SZ* is an extension of the recently published *BZ* algorithm. *BZ* scans the so-called "marriage table" to optimize global satisfaction and equity over all couples. It still gets about 5% unstable results in average. After a case study that sorts blocking situations into 4 types, we explain here how to resolve unstability in forcing blocking pairs to marry wrt. their type. *SZ* is compared to *BZ* and *Gale-Shapley* on 40000 instances of a 200 persons large population. An example of stereo reconstruction by *SZ* is given for illustration.

1 Introduction

In many computer vision applications including motion analysis or stereovision, matching is a crucial step towards accurate world understanding. Over the years, many matching methods were studied ranging from various kind of mere correlation up to relaxation or dynamic programming. In this paper we evaluate the "stable marriage" paradigm for image matching. That is a bi-partite graph optimisation technique for which the *BZ* algorithm was recently proposed, based on the so called *marriage table* representation [1]. As our application [2] is feature-pairing in robot vision − to stereo reconstruction, motion understanding or object recognition − there is an a priori equal importance of features from both sides. Then *fairness* (or *sex equality*) needed to be accounted for unlike in previously known algorithms where only stability matters. Moreover, matching had to be satisfactory whichever the correponding regions in an image pair thus requiring a *global satisfaction.* That meant for the algorithm to guarantee the best balanced solution among the many possible ones. And of course *stability* of the matching was fully considered. *BZ* provides matching results with maximum sex equality and global satisfaction, however about 5% of its found solutions are unstable. The present paper describes an improved version of *BZ* called *SZ* that grants stability by trading off with sex equality and global statisfaction.

The paper is organized as follows : we first revisit the *marriage table* representation and the *BZ* algorithm in section 1.1. Then we explain the origin of matching instability in section 1.2. In section 1.3 the extended algorithm, *SZ,* is proposed. Its performance is compared with the conventional *GS* algorithm's and the *BZ* algorithm's in section 1.4. Eventually, a level-line matching results for stereovison are shown for sake of illustration in section 1.5.

1.1 BZ algorithm revisited

The stable marriage problem was first studied by Gale and Shapley [3] and it is among the popular combinatorics problem [4] [5] [6] [7]. In this problem, two finite sub-sets M and W of two respective populations, say men and women, have to match. Assume n is the number of elements, $M = \{m_i\}_1^n$ and $W = \{w_j\}_1^n$. Each element x creates its preference list $l(x)$ i.e. it sorts all members of the opposite sex from most to less preferred (see example in the next section table 1). A matching $\mathcal{M}$ is a one to one correspondence between men and women. If (m, w) is a matched pair in $\mathcal{M}$, we note $\mathcal{M}(m) = w$ and $\mathcal{M}(w) = m$ and ρ_m is the rank of m in the list of w (resp. ρ_w the rank of w in the list of m) . Man m and woman w form a *blocking pair* if (m, w) is not in $\mathcal{M}$ but m prefers w to $\mathcal{M}(m)$ and w prefers m to $\mathcal{M}(w)$. Note that $(m, \mathcal{M}(m))$ and $(\mathcal{M}(w), w)$ are *blocked pairs.* The situation where (m, w) is blocking $(m, \mathcal{M}(m))$ and $(\mathcal{M}(w), w)$ is called *blocking situation.* If there is no blocking pair, then the marriage $\mathcal{M}$ is stable.

The *marriage table* is a representation of the stable marriages problem designed to meet the three criteria of stability, sex equality and global satisfaction. It is a table with $(n + 1)$ lines and $(n + 1)$ columns. Lines (resp. columns) frame the preference orders of men, $\{1 \cdots p \cdots N\ \infty\}$ (resp. women, $\{1 \cdots q \cdots N\ \infty\}$). The cell (p, q) contains pairs (m, w) such that w is the p^{th} choice of m, and m is the q^{th} choice of w. Cells can thus contain more than one pair or none. The cell (p, ∞) (resp. (∞, q)) contains the pairs (m, w) where w is the p^{th} choice of m (resp m the q^{th} choice of w) but m does not exist in her preference list (resp. w is not in his preference list).

So, the global satisfaction of matching can be defined by $\overline{S} = \sum_{(m,w)\in M}(\rho_m + \rho_w)$. Note that a solution with maximum global satisfaction would get matched pairs around the origin of the table (bottom-left). Conversely, sex equality tends to fit the diagonal of the marriage table. It is defined as $\overline{E} = \sum_{(m,w)\in M} |\rho_m - \rho_w|$.

The BZ algorithm consists of scanning the marriage table cells in order to first maximize both criteria concurrently. It scans anti-diagonals forward from maximum to minimum global satisfaction while each one is read in swinging from center to sides meaning maximum to minimum sex equality. In each cell, pairs are married if both partners are free. After all cells have been visited, the table is then revisited again to remove the blocking situations: a blocking pair gets married and corresponding blocked pairs are released. The process repeats until there is no more blocking situation (case "stability") or the iteration number is greater than the population size ("instability").

1.2 Matching instability

The instability of the marriages was studied by several authors [8] [9]. To understand where does matching instability come from let us unfold two matching examples on the same population (table 1). The table 2 shows an instance of the marrying process. For each repetition, free pairs are coupled and blocking situations removed. At step3 a stable matching is output. The table 3 shows a second instance of the same : free pairs to marry and blocking pairs to be removed are scrutinized in a different order. It can be noticed in this case that after marrying (m_1, w_1) to solve the blocking situation $-$ then releasing $w_3$and m_3- and later on marrying (m_3, w_3), the marriage at step4 turns again into the marriage at step1 . From then on the marriage process would *cycle* infinitely. Unlike with the previous scan, the current process could not find a stable solution and the matching will remain unstable for ever. In the *BZ* algorithm,

Man	Woman
$l(m_1) = w_2,\ w_1,\ w_3$	$l(w_1) = m_1,\ m_3,\ m_2$
$l(m_2) = w_1,\ w_3,\ w_2$	$l(w_1) = m_3,\ m_1,\ m_2$
$l(m_3) = w_1,\ w_2,\ w_3$	$l(w_1) = m_1,\ m_3,\ m_2$

Table 1. Preference list

Marriage	Blocking situation
$1 : (m_1, w_1)$	$(m_1, w_2) : [(m_1, w_1),\ (m_2, w_2)]$
$(m_2, w_2),\ (m_3, w_3)$	$(m_3, w_2) : [(m_3, w_3),\ (m_2, w_2)]$
$2 : (m_1, w_2)$	$(m_3, w_1) : [(m_3, w_3),\ (m_2, w_1)]$
$(m_2, w_1),\ (m_3, w_3)$	$(m_3, w_2) : [(m_3, w_3),\ (m_1, w_2)]$
$3 : (m_1, w_2)$	
$(m_2, w_3),\ (m_3, w_1)$	

Table 2. Example of a marriage without cycle

Marriage	Blocking situation
$1 : (m_1, w_1)$	$(m_1, w_2) : [(m_1, w_1),\ (m_2, w_2)]$
$(m_2, w_2),\ (m_3, w_3)$	$(m_3, w_2) : [(m_3, w_3),\ (m_2, w_2)]$
$2 : (m_1, w_2)$	$(m_3, w_1) : [(m_3, w_3),\ (m_2, w_1)]$
$(m_2, w_1),\ (m_3, w_3)$	$(m_3, w_2) : [(m_3, w_3),\ (m_1, w_2)]$
$3 : (m_1, w_3)$	$(m_1, w_1) : [(m_1, w_3),\ (m_2, w_1)]$
$(m_2, w_1),\ (m_3, w_2)$	$(m_3, w_1) : [(m_3, w_2),\ (m_2, w_1)]$
$4 : (m_1, w_3)$	$(m_1, w_1) : [(m_1, w_3),\ (m_3, w_1)]$
$(m_2, w_2),\ (m_3, w_1)$	$(m_1, w_2) : [(m_1, w_3),\ (m_2, w_2)]$

Table 3. Exemple of a marriage with cycle

marrying free pairs and cleaning blocking situations depends explicitly on the marriage-table scan that obeys independant constraints. In the next section, we classify all possible blocking situations to appear in a marriage table. The case list is further used to determining stable optimal marriages.

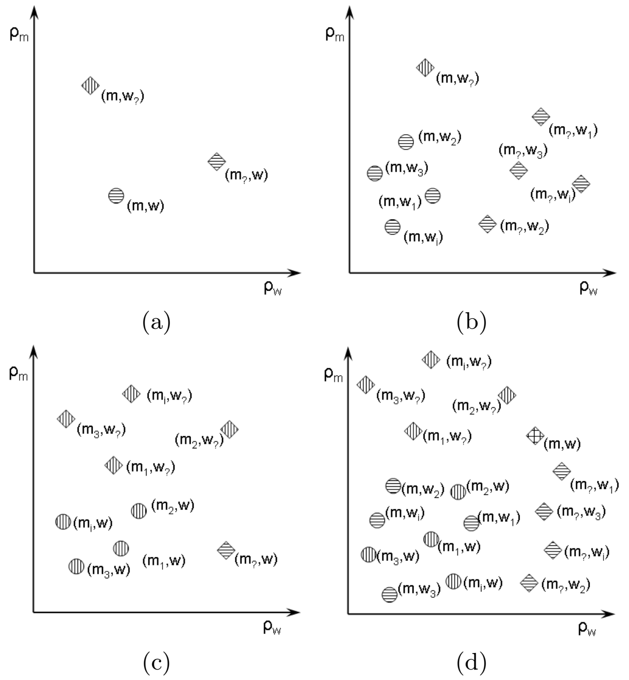

Fig. 1. Blocking situations in marriage table : (a) Type 1 (b)Type 2, (c) Type 3 (d) Type 4. Circles represent blocking pairs and squares represent blocked pairs. Vertical and horizontal lines feature the blocking situation by men and wowen respectively

Blocking situations and oscillating behaviours of marriages Not all blocking situations make cycles occur over the marriage table. We classify the blocking situations into four principal types :

Type1, already explained for the basics, comes from one blocking pair (m, w) which relates to one blocked pair by its man $(m, w_?)$ and one blocked pair by its woman $(m_?, w)$. (m, w) is not matched in $\mathcal{M}$ but m prefers w to $w_?$ and w prefers m to $m_?$. So, $w_?$ $(m_?)$ is the current choice of m (resp.w) in $\mathcal{M}$. The situation in the marriage table is displayed in figure 1(a)

Type2 is composed of M blocking pairs (m, w_j), $j = 1 \cdots M$ corresponding to one blocked pair by its man $(m, w_?)$ and M blocked pairs by their woman. $(m_?, w_j)$, $j = 1 \cdots M$. (m, w_j) are not matched in $\mathcal{M}$ but m prefers w_j to $w_?$ and w_j prefers m to $m_?$. The situation is shown in the figure 1(b) Type3 is a symmetric version of type2. It shows as N blocking pairs (m_i, w), $i = 1 \cdots N$ relating to one blocked pair by its woman $(m_?, w)$ and N blocked pairs by the man $(m_i, w_?)$, $i = 1 \cdots N$. (m_i, w) are not matched in $\mathcal{M}$ but m_i prefers w to $w_?$ and w prefers m_i to $m_?$. The situation is sketched figure1(c).

Type4 is a combination of 2 and 3. It results from $N + M$ blocking pairs (m_i, w), (m, w_j), where $i = 1 \cdots N$ and $j = 1 \cdots M$ generating N blocked pairs by the man $(m_i, w_?)$, $i = 1 \cdots N$, M blocked pairs by the woman $(m_?, w_j)$,

$j = 1 \cdots M$ and one pair blocked from both sides (m, w). The situation is that (m_i, w) and (m, w_j),where $i = 1 \cdots N$ and $j = 1 \cdots M$ are not matched in $\mathcal{M}$ but m_i prefers w to $w_?$ and w prefers m_i to $m_?$. And conversely, m prefers w_j to $w_?$ and w_j prefers m to $m_?$. The corresponding situation of the marriage table appears figure 1(d).

Cells of the marriage table are visited sequentially to marry free pairs, the scan generates dynamically blocking pairs to be eliminated after. Other scans, other marriages and then other blocking situations and pairs. So each instance of a blocking situation is independant in the sense that resolving it does not influence concurrent blocking situations at that step. Additionnally, we found over 40000 tries that early blocking pairs tend to be majoritarily of type1. As the process goes on, instances belong more and more to the three other types while decreasing in number. That is trivially explained by the logics of the situation:

[$(m, w_?)$got married first] $\Longrightarrow$ [m and $w_?$were free]
then[(m, w)blocks $(m, w_?)$]**and**[$(m_?, w)$got married]
$\Longrightarrow$ [$m_?$was free **and** $w$was free because $m$was married]
then[(m, w)gets married]
and [$(m, w_?)$and $(m_?, w)$are released]
$\Longrightarrow$ $w_?$ and $m_?$ will marry further in the scan

The longer the process the longer distance between blocking and blocked pairs and the more likely multiple blocking pairs for same blocked pairs.

Reaching stability, the process stops when there are no more blocking pairs in the table. Failing to do so (unstable marriage) corresponds to *oscillations* of blocking pairs and *cycles* of the process. Cycles cross the diagonal of the marriage table (equal satisfaction), possibly several times, corresponding to oscillations of blocking pairs from "man satisfaction side" to "woman satisfaction side" and back.

Oscilation and cycle elimination For the matching solution to become stable oscillations and cycles have to be eliminated. Although they are not independant phenomenas we can get rid of the oscillation and the cycle separately.

Stopping oscilations that relate to sex equality can be done by forcing man-first or woman-first matching when marrying the free pairs.We try both, and keep the best as for satisfaction and equity.

Eliminating cycles that relate to the stack of candidates for a same person could be performed by marrying an optimal blocking pair among the several possible ones. It applies for each situation type separately. In type 1 there is only one blocking pair for the optimal choice. Type 2, all blocking pairs contain the same man m with different partners w_j. Ideally we would choose the blocking pair (m, w^*) which contains the woman $w^* \in \{w_j\}$ that m prefers the most (that is man-first).

$$(m, w)^* = \left\{(m, w^*) \diagup \rho_{w^*} \ll_m \rho_{w_j}, j = 1, \cdots M\right\} \quad (1)$$

In reality choosing the blocking pair that features the woman w_j who prefers m (and the one prefered by m in case of equality i.e. the first couple met along a vertical scan) succeeds as well. Since oscillation removal does not allow to switch from man-first to woman-first or conversely during a same process, actually we keep the one retained for stopping oscillations.

Type 3 is symmetric of type 2, $(m, w)^*$ ideally contains the man $m^* \in \{m_i\}$ that w prefers the most.

$$(m, w)^* = \{(m^*, w) / \rho_{m^*} \ll_w \rho_{m_i}, i = 1, \cdots, N\} \tag{2}$$

Here again the selection policy depends on oscillation removal and is maintained all along the elimination process.

Type 4 combines types 2 and 3. There are two different groups of blocking pairs then two possible optimal choices, respectively man-first (eq.1) and women-first (eq.2) 1(d). We keep again the option for oscillation removal.

In the next section we propose a stable-marriages algorithm designed from the criteria introduced in this section.

1.3 New algorithm : Stable Zigzag (SZ) with man-optimal (SZ_m) or woman-optimal (SZ_w)

The so called *Stable Zigzag* algorithm summarizes the study above . It runs in two separate phases : it first applies the *BZ* procedure aiming to find the best solution as for global satisfaction, sex equality and then stability. As previously mentioned (section 1) that rules about 95% instances. The second phase deals with oscillation/cycle removal for marriages to get stable. It concerns the 5% unstable marriages output by *BZ*. Two options in this case : *SZ* with man-optimal (SZ_m) and *SZ* with woman-optimal (SZ_w).

SZ_m gives the scan priority to the man side : blocking situations are searched for and typified by scanning rows of the table bottom-up and left-right, then pairs will be married upon release iteratively according to the following :

- Type 1, marry the blocking pair (there is only one) and release related blocked pairs as shown figure 2(a).
- Type 2, marry the blocking pair which contains the woman $w^* \in \{w_j\}$, $j = 1, \cdots, M$ that m prefers among $\{w_j\}$and then release corresponding blocked pairs , see figure 2(b).
- Type 3, marry the blocking pair which contains the man $m^* \in \{m_i\}$, $i = 1, \cdots, N$ who prefers w (the first couple met along the present horizontal scan in case of equality of preferences) and then release the tied blocked pairs as in figure 2(c).
- Type 4, is strictly the same as type 2, figure 2(d).

SZ_w is the exact symmetric version of the former, in giving the importance to the woman side : it marries the free pairs by scanning columns and women chose first in a systematic manner.

Algo. 1 shows SZ_w. To get the SZ_m procedure it is enough to exchange m and w sytematically, in keeping indexes and exponents.

Algorithm 1: Stable Zigzag with woman-optimal

```
begin
    Run BZ algorithm
    while there is cycle, blocking situtation do
        foreach column : bottom -up do
            foreach pair (m, w) do
                if m and w are free then
                    Marry m with w
        foreach column : bottom-up do
            foreach pair (m, w) do
                if (m, w) is blocking pair then
                    Identify situation type
                    Type 1 : (m, w)* = (m, w)
                    Type 2 : (m, w)* = {(m, w*) ⁄ ρ_m ≪_{w*} (ρ_m ≪_{w_i})}
                    Type 3 : (m, w)* = {(m*, w) ⁄ ρ_{m*} ≪_w ρ_{m_i}}
                    Type 4 : (m, w)* = {(m*, w*) ⁄ ρ_{m*} ≪_w ρ_{m_i}}
        foreach colom : bottom-up do
            foreach pair (m, w) do
                if (m, w) is (m, w)* then
                    Release the blocked pairs : (m, w_?) and (m_?, w)
                    Marry m with w
end
```

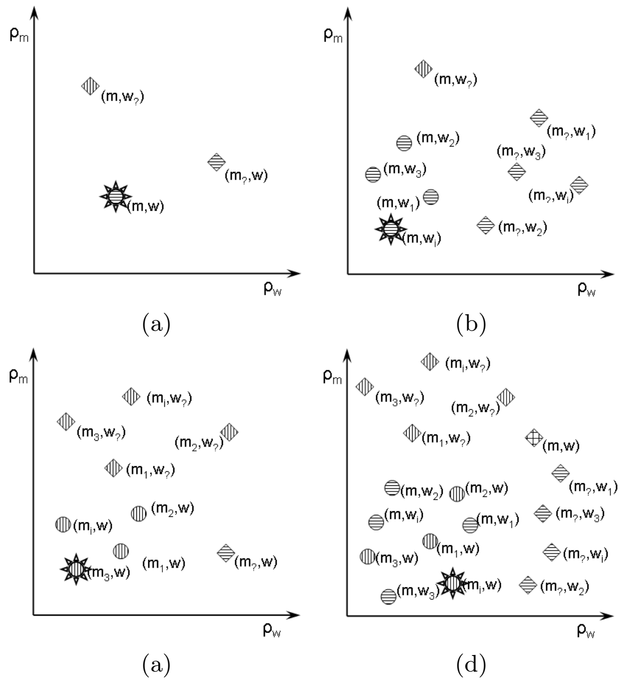

Fig. 2. Elimination of a blocking situation by *SZ* with man-optimal : (a) type 1, (b) type 2, (c) type 3, (d) type 4. The star shows the selected blocking pair in every case.

1.4 Algorithm performance

We studied experimentally the global satisfaction, sex equality and stability issued by *SZ* and compared with *GS* (Gale-Shapley) and *BZ*. About 40.000 instances − couples of preference list sets − were built at random for a 200 persons large population. Each algorithm is run and results are displayed. From the experiments, we first note that the results by *SZ* are stable for all instances while around 5% results by *BZ* were unstable. The global satisfaction is shown figure 3, it appears that its form keeps some similarity by subsets of 30 in average. Figure 4 zooms on 30 instances. Let us define the number b of instances where *SZ* is better than *GS* and *BZ* as

$$\begin{array}{c} b = \sum_{\text{all instances}} Y_{[GS_{m,w}(BZ) - SZ_{m,w}]} \text{with} \\ Y_x = 1 \text{ if } x > 0 \\ Y_x = 0 \;\text{ else} \\ \text{and } \beta = \frac{100 \times b}{\text{number\_of\_instances}} \end{array} \qquad (3)$$

Comparing $SZ_{m,w}$ with BZ as for the global satisfaction, $\beta = 72\%$ and 72.8% for m and w respectively. Same way, comparing $SZ_{m,w}$ with $GS_{m,w}$, $\beta = 100\%$ in both cases. *SZ* is always better than *GS*, figure5.

For sex equality, comparing $SZ_{m,w}$ with BZ, $\beta = 58.4\%$ and 56% respectively, figure 6. And comparing $SZ_{m,w}$ with $GS_{m,w}$, $\beta = 100\%$ in both cases , figure 7.

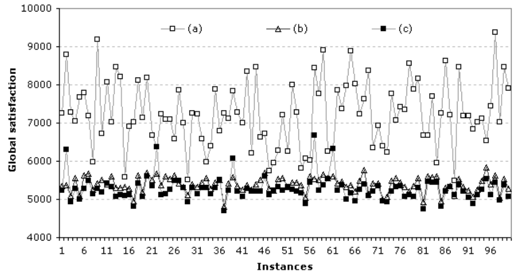

Fig. 3. Comparing global satisfaction between methods : (a) the better GS (b) BZ algorithm (c) the better SZ

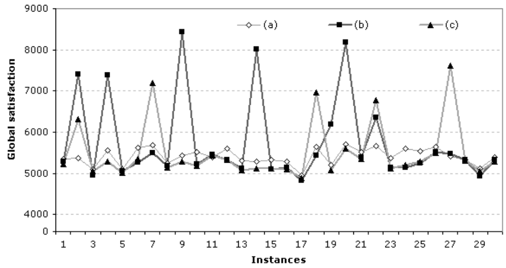

Fig. 4. Close up on global satisfaction between methods (30 instances) : (a) BZ algorithm (b) SZ_m (c) SZ_w

Note that BZ performs better than SZ in cases where SZ_m and SZ_w score very differently. Then BZ performs better than one SZ , whether it is for statisfaction or sex equality and the distance between them is important too: $\frac{S_{SZ}-S_{BZ}}{S_{SZ}}$ = 18.81% in average for the m option and 21.28% for w , and $\frac{E_{SZ}-E_{BZ}}{E_{SZ}} = 32.16\%$ in average for the m option and 33.17% for w . But in the many cases where it is the contrary the improvement from SZ is not real significant ($\frac{S_{BZ}-S_{SZ}}{S_{BZ}}$ =3.17% in average for the m option and 3.15% for w and $\frac{E_{BZ}-E_{SZ}}{E_{BZ}}$ =3.41% in average for the m option and 3.48% for w). As a last index, one can compare the following means: average difference in satisfaction when BZ outperforms $SZ, \partial^S_{B,S} = 1258$, average difference in satisfaction when SZ outperforms BZ, $\partial^S_{S,B}$= 166, average difference in equity when BZ outperforms $SZ, \partial^E_{B,S} = 1298$, average difference in equity when SZ outperforms $BZ, \partial^E_{B,S} = 95$.

Eventually it is important to keep in mind that if SZ is globally better than BZ and GS, its complexity in the $(O(n^4))$ is high compared to BZ $(O(n^3))$ and GS $(O(n^2))$ respectively.

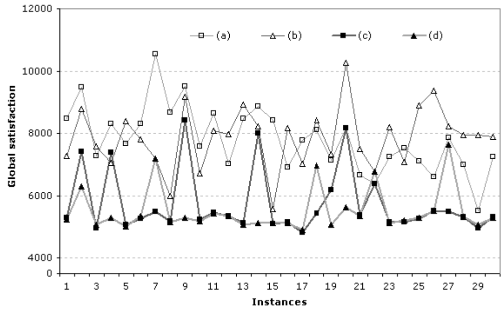

Fig. 5. Comparing global satisfaction between methods : (a) GS_m (b) GS_w (c) SZ_m (d) SZ_w

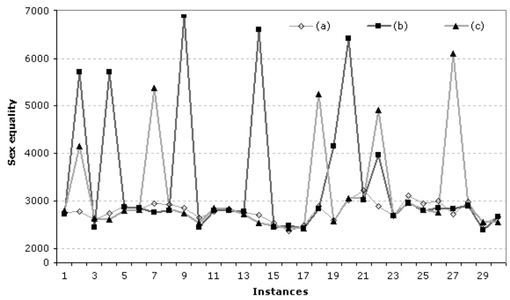

Fig. 6. Comparing sex equality between methods : (a) BZ algorithm (b) SZ_m (c) SZ_w

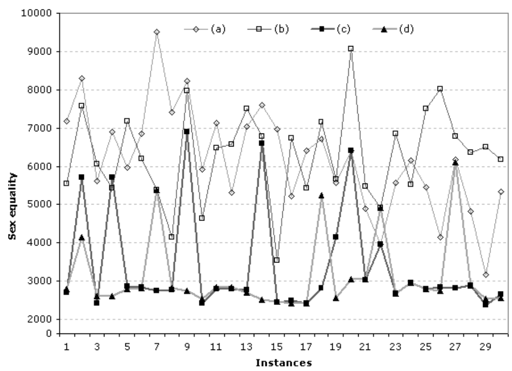

Fig. 7. Comparing sex equality between methods : (a) GS_m (b) GS_w (c) SZ_m (d) SZ_w

1.5 Experiments

For sake of illustration, we show here one example in stereovison. The matching relies on level-line junctions. Junctions are extracted as features from left and right images separately. Each primitive selects prefered mates in the other image, and then sorts them from most to less preferred according to junctions similarity. Then *SZ* is executed. Figure 8(a) and (b) show the stereo pair. Extracted junctions are displayed in figure 8(c) and (d) respectively. Matching results, before and after outliers removal, are in figure 8(e) and (f) respectively.

1.6 Conclusion

In this paper, we proposed the so called *SZ* stable marriage algorithm. It relies on scans of the *marriage table* to optimize the global satisfaction and the sex equality, and then to remove blocking pairs if necessary. Its performance in terms of global satisfaction and sex equality is compared with that of *GS* and *BZ*. While stability of the matching is fully supported − the important progress from the *BZ* algorithm − both satisfaction and equity are increased in 60 to 70% of the processed cases. However, when they drop it is in a proportion of about half the *BZ* score. Moreover complexity grows from $O(n^3)$ to $O(n^4)$.

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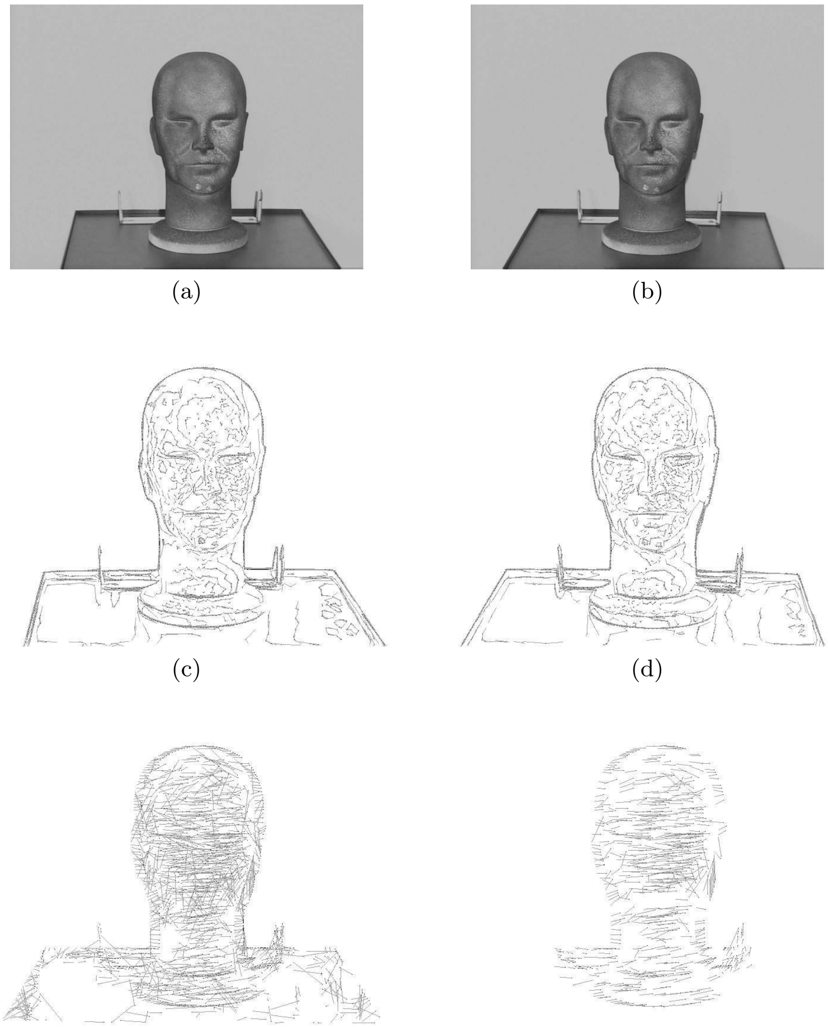

(a) (b)

(c) (d)

(e) (f)

Fig. 8. Stereo matching : (a)(b) Original images (c)(d) Extracted level-lines junctions (e)(f) Matching results